

The Orbifolds of Permutation-Type as Physical String Systems at Multiples of $c = 26$ V. Cyclic Permutation Orbifolds

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Abstract

I consider the \mathbb{Z}_λ , λ prime free-bosonic permutation orbifolds as interacting physical string systems at $\hat{c} = 26\lambda$. As a first step, I introduce twisted tree diagrams which confirm at the interacting level that the physical spectrum of each twisted sector is equivalent to that of an ordinary $c = 26$ closed string. The untwisted sectors are surprisingly more difficult to understand, and there are subtleties in the sewing of the loops, but I am able to propose provisional forms for the full modular-invariant cosmological constants and one-loop diagrams with insertions.

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1 Introduction

At the level of examples, current-algebraic conformal field theory [1,2] and orbifold theory [3-8] are almost as old as string theory itself [9-12]. It is only in the last few years however that the orbifold program [13-23,24-27] has in large part completed the local description of the general closed- and open-string orbifold conformal field theory. See Refs. [22,28] for short reviews of the orbifold program and related recent developments in orbifold theory.

Extending the results of the orbifold program, I have recently proposed [28-31] that the *orbifolds of permutation-type* define generically-new physical open- and closed-string systems at multiples of conventional critical central charges. The simplest examples of these systems are the free-bosonic cases at $\hat{c} = 26K$, for which we have so far studied the following topics:

I. The extended action formulations of the twisted sectors of these orbifolds, which show *new permutation-twisted world-sheet gravities* [28]. The extended diffeomorphism invariances and extended, twisted Virasoro constraints of these actions indicate that, as string theories, the orbifolds of permutation-type can be free of negative-norm states. The principles discussed in this reference will suffice to construct the twisted world-sheet supergravities of the corresponding superstring orbifolds of permutation-type.

II. The twisted reparametrization ghosts and *new twisted BRST systems* [29] of \mathbb{Z}_2 -twisted permutation gravity, which imply the *extended physical state conditions* (extended, twisted Virasoro constraints) of all $\hat{c} = 52$ matter. Beyond this simple case, the twisted BRST systems of the general world-sheet permutation gravities have not yet been worked out.

III. The extended Virasoro generators and *physical spectra* of all twisted $\hat{c} = 52$ strings [30], including an equivalent, unconventionally-twisted $c = 26$ description of their spectra. This analysis provides further evidence that the orbifold-string systems of permutation-type can be free of negative-norm states, and moreover shows that a few of the simplest (half-integer moded) open and closed $\hat{c} = 52$ strings have the same spectra as ordinary untwisted $c = 26$ strings. Beyond these simple cases, the twisted $\hat{c} = 52$ strings are apparently new.

IV. *Orientation orbifolds include orientifolds* [27]. The orientation-orbifold

string systems [28-30] involve dividing the closed string by automorphism groups which contain world-sheet orientation-reversing automorphisms. They therefore contain an equal number of generically-twisted closed- and open-string sectors, each of the latter at $\hat{c} = 52$. In this fourth paper of the series, twisted tree diagrams are constructed to study a particular example with a single half-integer moded open-string sector, and evaluation of the trees confirms [30] at the interacting level that this orientation-orbifold system is nothing but the archetypal orientifold in disguise. There are many other free-bosonic orientation orbifolds (with higher fractional modeing) which are not equivalent [29] to conventional orientifolds, but these “generalized orientifolds” have not yet been studied at the interacting level.

In this fifth paper of the series, I return to the closed-string permutation orbifolds at central charge $\hat{c} = 26K$, whose general form

$$\frac{U(1)^{26K}}{H_+}, \quad H_+ = H(\text{perm}) \times H, \quad H(\text{perm}) \subset S_K \quad (1.1)$$

includes the uniform action of automorphisms H on each untwisted closed-string copy $U(1)^{26}$. For simplicity, the discussion of these orbifolds is continued here only for the case of continuous (decompactified) zero modes, but this can be straightforwardly generalized. The physical spectrum of all the twisted sectors at $\hat{c} = 52$ (all H_+ with $K = 2$ and $H(\text{perm}) = \mathbb{Z}_2$) was studied in Ref. [30], where it was noted in particular that the spectrum of the twisted sector with trivial H (the ordinary \mathbb{Z}_2 -permutation orbifold) is equivalent to that of an ordinary untwisted $c = 26$ closed string. Generalizing the \mathbb{Z}_2 orbifold, I will restrict the discussion here to the string theories of the simplest *cyclic permutation orbifolds*

$$\frac{U(1)^{26\lambda}}{\mathbb{Z}_\lambda}, \quad \lambda \text{ prime}, \quad \hat{c} = 26\lambda \quad (1.2)$$

but I will consider this family at the *interacting level* through one loop.

As a first step, I introduce twisted tree diagrams (see Sec. 2) for these orbifolds which confirm at the interacting level for all prime λ our earlier conclusion [30] for the free theory at $\lambda = 2$: The physical spectrum of each of the $\lambda - 1$ twisted sectors of $U(1)^{26\lambda}/\mathbb{Z}_\lambda$ is equivalent to that of an ordinary untwisted $c = 26$ string. As emphasized in Refs. [30,31], this so-called *spectral equivalence* holds *only* for the string amplitudes, which involve extra integrations over the correlators of the orbifold CFT’s.

Sewing the twisted trees gives an integrated form of the standard [7] twisted contribution to the orbifold partition function, and the following formal integration identity (see Eq. 3.10)

$$\int_{F_N} \frac{d^2\tau}{(\text{Im}\tau)^2} \left(Z(\tau, \bar{\tau}) - \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right) = 0 \quad (1.3a)$$

$$F_N : |\text{Re}\tau| \leq \frac{1}{2}, \text{Im}\tau \geq 0 \quad (1.3b)$$

then verifies the same spectral equivalence on the torus. Here $Z(\tau, \bar{\tau})$ is the partition function of the ordinary untwisted closed string $U(1)^{26}$, and F_N is the *naive* integration range encountered in the *naive (operator) sewing* [31,9] of any string loop. This operator sewing is discussed first for the orbifold cosmological constant in Subsecs 3.2 and 3.3, where it is called summing over the *particle-theoretic content* of the theory, and later for the loop with insertions in Sec. 6.

The particle-theoretic contributions of the untwisted sector are surprisingly more difficult to understand: I had expected to find included here the non-linear untwisted contribution $Z^\lambda(\tau, \bar{\tau})$ of the standard orbifold partition function [6], but in fact (see Subsec. 4.1) any such non-linear contribution is incompatible with one-loop structure in string theory!

Indeed I argue in Subsec. 4.2 and Sec. 6 that the particle-theoretic contribution of the untwisted sector to the sewn loop is equivalent to that of λ ordinary untwisted closed strings, which is also consistent with the cross-channel behavior of the twisted trees. Moreover, according to a second formal integration identity (see Eq. (4.8))

$$\int_{F_N} \frac{d^2\tau}{(\text{Im}\tau)^2} \left(\lambda Z(\tau, \bar{\tau}) - Z(\lambda\tau, \lambda\bar{\tau}) \right) = 0 \quad (1.4)$$

this situation can be equivalently described by keeping only the so-called *diagonal contribution* $Z(\lambda\tau, \lambda\bar{\tau})$ to the untwisted sector (see Refs. [6] and [9]).

Secs. 5 and 6 finally assemble our observations about the twisted and untwisted loop contributions. The central point here is that the formal integration identities (1.3) and (1.4) generate certain *ambiguities* in the naive

or particle-theoretic form of the loops, which persist in the corresponding modular-invariant forms. Thus I am able to propose for future study the following one-parameter β -family of provisional forms for the full modular-invariant cosmological constants (see Eq. (5.1))

$$(\alpha'_c)^{13}\Lambda^{(\lambda)}(\beta) = -\frac{1}{2} \int_F \frac{d^2\tau}{(\text{Im}\tau)^2} \left\{ (2\lambda - 1 - (\lambda + 1)\beta)Z(\tau, \bar{\tau}) + \right. \\ \left. + \beta \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \right\} \quad (1.5a)$$

$$F : |\text{Re}\tau| \leq \frac{1}{2}, \text{Im}\tau \geq 0, |\tau| \geq 1 \quad (1.5b)$$

where F is the standard fundamental region of the modular group. In these provisional forms, the parameter β describes the ambiguity in the naive sewing of the loops. The corresponding results for the one-loop diagrams with insertions are given in Eq. (6.1).

I emphasize that the simplest member of this family is $\beta = 0$

$$(\alpha'_c)^{13}\Lambda^{(\lambda)} = -\frac{2\lambda - 1}{2} \int_{F_N} \frac{d^2\tau}{(\text{Im}\tau)^2} Z(\tau, \bar{\tau}) \quad (1.6)$$

which is clearly equivalent to the contribution of λ ordinary $c = 26$ strings from the untwisted sector and $\lambda - 1$ ordinary $c = 26$ strings from the twisted sectors. Other interesting special cases are discussed in Subsec. 5.2 but, since all values of β are formally associated to the *same* naive sewing of the trees, the entire β -family should be further tested against other string- and conformal-field-theoretic intuitions.

In this connection, Subsec. 5.2 also includes some remarks (see in particular Eq. (5.9)) on hitherto-unnoticed *modular-covariant subsets* of permutation-orbifold characters – which exclude the so-called off-diagonal characters [10] of the untwisted sectors. These closed subsets are the natural infrastructure for partition functions of the form (1.5) with no non-linear term $Z^\lambda(\tau, \bar{\tau})$, and they may be useful in deciding among the provisional forms.

2 The Twisted Sectors on the Sphere

2.1 The Twisted Algebras of Sector σ

Before considering their corresponding string theories, I collect here some useful facts about our class of orbifolds as *conformal field theories*.

The cyclic permutation orbifold [13-15,17-19,21]

$$\frac{U(1)^{26\lambda}}{\mathbb{Z}_\lambda}, \quad \lambda \text{ prime, } \hat{c} = 26\lambda \quad (2.1)$$

has $\lambda - 1$ identical twisted sectors $\sigma = 1, \dots, \lambda - 1$ at central charge 26λ . In each twisted sector, one finds the σ -independent (left-mover) extended Virasoro generators and twisted algebras [13]:

$$\begin{aligned} \hat{L}_r(m + \frac{r}{\lambda}) &= \frac{13}{12} \left(\lambda - \frac{1}{\lambda} \right) \delta_{m+\frac{r}{\lambda},0} \\ &\quad - \frac{\eta^{\mu\nu}}{2\lambda} \sum_{s=0}^{\lambda-1} \sum_p \circ \hat{J}_{s\mu}(p + \frac{s}{\lambda}) \hat{J}_{r-s,\nu}(m - p + \frac{r-s}{\lambda}) \circ_M \end{aligned} \quad (2.2a)$$

$$[\hat{J}_{r\mu}(m + \frac{r}{\lambda}), \hat{J}_{s\nu}(n + \frac{s}{\lambda})] = -\lambda \eta_{\mu\nu} (m + \frac{r}{\lambda}) \delta_{m+n+\frac{r+s}{\lambda},0} \quad (2.2b)$$

$$[\hat{L}_r(m + \frac{r}{\lambda}), \hat{J}_{s\mu}(n + \frac{s}{\lambda})] = -(n + \frac{s}{\lambda}) \hat{J}_{r+s,\mu}(m + n + \frac{r+s}{\lambda}) \quad (2.2c)$$

$$\begin{aligned} [\hat{L}_r(m + \frac{r}{\lambda}), \hat{L}_s(n + \frac{s}{\lambda})] &= (m - n + \frac{r-s}{\lambda}) \hat{L}_{r+s}(m + n + \frac{r+s}{\lambda}) \\ &\quad + \frac{26\lambda}{12} (m + \frac{r}{\lambda}) ((m + \frac{r}{\lambda})^2 - 1) \delta_{m+n+\frac{r+s}{\lambda},0} \end{aligned} \quad (2.2d)$$

$$\eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix}, \quad \mu, \nu = 0, 1, \dots, 25, \quad \bar{r}, \bar{s} = 0, 1, \dots, \lambda - 1. \quad (2.2e)$$

Here η is the Minkowskian target-space metric of the ordinary critical closed string $U(1)^{26}$, $\circ \cdot \circ_M$ is standard mode normal ordering [13-15,17-19,21] and Eq. (2.2d) is an orbifold Virasoro algebra [13,21,29,32] of order λ . The physical Virasoro generators $\{\hat{L}_0(m)\}$ of each sector satisfy the implied integral Virasoro subalgebra at $\hat{c} = 26\lambda$.

Including the twisted vertex operators $\hat{g}(\mathcal{T}) = :exp(i\mathcal{T} \cdot \hat{x}):$ of sector σ , one finds the additional algebras [15,16,18] :

$$[\hat{J}_{r\mu}(m + \frac{r}{\lambda}), \hat{g}(\mathcal{T}, \bar{z}, z)] = t_r T_\mu z^{m+\frac{r}{\lambda}} \hat{g}(\mathcal{T}, \bar{z}, z) \quad (2.3a)$$

$$[\hat{L}_r(m + \frac{r}{\lambda}), \hat{g}(\mathcal{T}, \bar{z}, z)] = t_r z^{m+\frac{r}{\lambda}} (z\partial_z + (m + \frac{r}{\lambda} + 1)) \Delta(T) \hat{g}(\mathcal{T}, \bar{z}, z) \quad (2.3b)$$

$$[T_\mu, \hat{g}(\mathcal{T}, \bar{z}, z)] = [t_r, \hat{g}(\mathcal{T}, \bar{z}, z)] = 0 \quad (2.3c)$$

$$\mathcal{T}_{r\mu} = t_r T_\mu, \quad [T_\mu, T_\nu] = 0, \quad \Delta(T) = -\frac{T^2}{2} \quad (2.3d)$$

$$(t_r)_s^t = \delta_{r+s-t, \text{omod } \lambda}, \quad t_r t_s = t_{r+s}, \quad t_0 = \mathbb{1}_\lambda. \quad (2.3e)$$

In these relations, $\{T\}$ are the dimensionless momenta (for simplicity assumed here to be continuous) and \hat{g} has the same $\lambda \times \lambda$ matrix structure as the matrices $\{t\}$. There are also right-mover copies $\{\hat{\hat{J}}_{r\mu}\}$ and $\{\hat{\hat{L}}_r\}$ in each twisted sector with the same algebra (2.2) and $z \rightarrow \bar{z}$ in the coefficients of Eqs. (2.3a,b). The explicit form and diagonal monodromy of the twisted vertex operators¹ of all free-bosonic orbifolds are given in Ref. [16], and a concrete example will be considered in Subsec. 2.4.

We will also need the momentum-boosted twist-field state $|\mathcal{T}\rangle$ of sector σ which satisfies [18,19,21]

$$|\mathcal{T}\rangle = |T\rangle \mathbb{1}_\lambda = \lim_{z \rightarrow 0} |z|^{2\Delta(T)(1-\lambda^{-1})} \hat{g}(\mathcal{T}, \bar{z}, z) |0\rangle \quad (2.4a)$$

$$\hat{J}_{r\mu}((m + \frac{r}{\lambda}) \geq 0) |T\rangle = \delta_{m+\frac{r}{\lambda}, 0} |T\rangle T_\mu \quad (2.4b)$$

$$\hat{L}_r((m + \frac{r}{\lambda}) \geq 0) |T\rangle = \delta_{m+\frac{r}{\lambda}, 0} |T\rangle \left(\frac{13}{12} \left(\lambda - \frac{1}{\lambda} \right) + \frac{\Delta(T)}{\lambda} \right) \quad (2.4c)$$

as well as right-mover copies of Eqs. (2.4b,c). Note that the state $|T\rangle$ in these relations is a state in the oscillator Hilbert space, while the state $|\mathcal{T}\rangle$ has an additional (trivial) matrix structure.

¹For simplicity I have here taken $T \rightarrow -T$ in the right-mover vertex operators of Ref. [19], which uniformizes the right- and left- mover signs in Eqs. (2.3a,b) without changing any orbifold correlators.

2.2 The Twisted Trees of Sector σ

I begin the construction of the permutation-orbifold *string systems* by defining the *string propagator* in twisted sector σ :

$$\hat{D}[\hat{L}_0(0), \hat{\bar{L}}_0(0)] \equiv \frac{\delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0)}}{\lambda(\hat{L}_0(0) + \hat{\bar{L}}_0(0) - 2\hat{a}_\lambda)} \quad (2.5a)$$

$$\hat{a}_\lambda \equiv \frac{13\lambda^2 - 1}{12\lambda}. \quad (2.5b)$$

The factor in the numerator of \hat{D} is ordinary Kronecker delta and, according to the twisted BRST system of Ref. [29], the quantity $\hat{a}_2 = 17/8$ is universal for all $\hat{c} = 52$ matter. The form of \hat{a}_λ for $\lambda \geq 3$ was conjectured in Ref. [30], and we shall check the consistency of this conjecture in the following subsection. Useful integral representations of the propagator include

$$\begin{aligned} \hat{D}[\hat{L}_0(0), \hat{\bar{L}}_0(0)] &= \int_{|z| \leq 1} \frac{d^2 z}{2\pi\lambda} z^{\hat{L}_0(0)} \bar{z}^{\hat{\bar{L}}_0(0)} |z|^{-\frac{(13\lambda-1)(\lambda+1)}{6\lambda}} \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} e^{2\pi i r (\hat{L}_0(0) - \hat{\bar{L}}_0(0))} \\ &= \int_{|z| \geq 1} \frac{d^2 z}{2\pi\lambda} z^{-\hat{L}_0(0)} \bar{z}^{-\hat{\bar{L}}_0(0)} |z|^{\frac{(13\lambda+1)(\lambda-1)}{6\lambda}} \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} e^{-2\pi i r (\hat{L}_0(0) - \hat{\bar{L}}_0(0))} \end{aligned} \quad (2.6)$$

where the sums set $\hat{\bar{L}}_0(0) \simeq \hat{L}_0(0)$ modulo the integers.

The conformal weight of the orbifold-string ground state $|T\rangle$ and its twisted vertex operator $\hat{g}(T)$ is then determined

$$\Delta(T) = \lambda \left(\hat{a}_\lambda - \frac{13}{12} \left(\lambda - \frac{1}{\lambda} \right) \right) = 1, \quad T^2 = -2 \quad (2.7a)$$

$$\left(\hat{L}_r \left(\left(m + \frac{r}{\lambda} \right) \geq 0 \right) - \hat{a}_\lambda \delta_{m+\frac{r}{\lambda}, 0} \right) |T\rangle = 0 \quad (2.7b)$$

by comparing the poles of the propagator to Eq. (2.4c). In this case, the vertex operator “boost” relation

$$\hat{g}(T, \bar{z}, z) = z^{\hat{L}_0(0)} \bar{z}^{\hat{\bar{L}}_0(0)} \hat{g}(T, 1, 1) z^{-\hat{L}_0(0)-1} \bar{z}^{-\hat{\bar{L}}_0(0)-1} \quad (2.8)$$

follows from Eq. (2.3b).

We are now ready to define the (σ -independent) n -point *twisted tree graphs* of each twisted sector σ of the orbifold-string system by the so-called sidewise construction [33,3,12,31]

$$\hat{A}_n^{(\lambda)}(\{\mathcal{T}\}) \equiv S^T \langle -T^{(n)} | \hat{g}(\mathcal{T}^{(n-1)}, 1, 1) \hat{D}[\hat{L}_0(0), \hat{\bar{L}}_0(0)] \hat{g}(\mathcal{T}^{(n-2)}, 1, 1) \dots \hat{D}[\hat{L}_0(0), \hat{\bar{L}}_0(0)] \hat{g}(\mathcal{T}^{(2)}, 1, 1) | T^{(1)} \rangle S + \dots \quad (2.9a)$$

$$S_r = (S^T)^r = \frac{1}{\sqrt{\lambda}}, \quad \bar{r} = 0, 1, \dots, \lambda - 1, \quad T^2 = -2 \quad (2.9b)$$

which shows any particular twisted $\hat{c} = 26\lambda$ sector running horizontally (side-wise). Here the product of the twisted vertex operators is defined by matrix multiplication² with all $\{t_r^i = t_r\}$, and the final ellipsis denotes symmetrization with respect to the arguments of the vertex operators. Finally, note that the normalized “spinor” S in Eq. 2.9b is the simultaneous eigenvector of the t matrices

$$S^T S = 1, \quad t_r S = \mathbb{1}_\lambda S, \quad S^T t_r = S^T \mathbb{1}_\lambda, \quad \bar{r} = 0, \dots, \lambda - 1 \quad (2.10)$$

where I remind that $t_0 = \mathbb{1}_\lambda$ is the $\lambda \times \lambda$ unit matrix (see Eq. (2.3e)). This completes my definition of the twisted trees, whose properties will now be studied in some detail.

I turn first to some properties of the external particles emitted from the twisted sectors (channels) of these tree graphs, which will allow us to understand the hidden cross channel of the twisted trees as the *untwisted channel* of each cyclic permutation orbifold.

In this connection, note that the properties (2.10) of the spinor S allow us to write the vertex operator on S in a variety of ways

$$\hat{g}(\mathcal{T})S = \hat{\hat{g}}(T)S = \hat{g}_I(T)S, \quad I = 0, 1, \dots, \lambda - 1 \quad (2.11a)$$

$$\hat{g}(\mathcal{T}) \sim \circ \exp(i \sum_{r\mu} t_r T_\mu \hat{x}_{r\mu}) \circ \quad (2.11b)$$

$$\hat{\hat{g}}_I(T) \sim \circ \exp(it_I \sum_{r\mu} T_\mu \hat{x}_{r\mu}) \circ \quad (2.11c)$$

$$\mathbb{1}_\lambda \hat{g}_0(T) = \mathbb{1}_\lambda \hat{\hat{g}}(T) \quad (2.11d)$$

$$\hat{\hat{g}}(T) = \hat{g}(\mathcal{T} \rightarrow T) \sim \circ \exp(i \sum_{r\mu} T_\mu \hat{x}_{r\mu}) \circ \quad (2.11e)$$

²This choice seems necessary to obtain the extended Ward identities of the following subsection.

and these properties imply the following identities on the twisted trees:

$$\hat{A}_n^{(\lambda)}(\{\mathcal{T}\}) = \hat{A}_n^{(\lambda)}(\{T\})_I, \quad I = 0, 1, \dots, \lambda - 1 \quad (2.12a)$$

$$\equiv S^T \langle -T^{(n)} | \hat{g}_I \hat{D} \hat{g}_I \dots \hat{g}_I \hat{D} \hat{g}_I | T^{(1)} \rangle S + \dots \quad (2.12b)$$

$$= \langle -T^{(n)} | \hat{g} \hat{D} \hat{g} \dots \hat{g} \hat{D} \hat{g} | T^{(1)} \rangle + \dots \quad (2.12c)$$

The equivalences (2.11) and (2.12) allow us to interpret the vertex operator \hat{g}_I as the *emission vertex* for a tachyon ($T^2 = -2$) of type $I = 0, 1, \dots, \lambda - 1$ from each twisted channel $\sigma = 1, \dots, \lambda - 1$. Each of the tachyon types is in fact spinless and untwisted because the amplitude $\hat{A}_I = \hat{A}$ for emission of many tachyons of type I has no target-space or world-sheet indices. The I -independence $\hat{A}_I = \hat{A}$ of the multiple-emission amplitudes also tells us that all λ of these tachyons are identical because they couple identically to each of the $\lambda - 1$ twisted channels. This set of λ identical untwisted tachyons is then exactly what is expected (before symmetrization on $\{I\}$) in the *untwisted sector* (cross channel) of the orbifold $U(1)^{26\lambda}/\mathbb{Z}_\lambda$. These remarks are illustrated in Fig. 1, where the external untwisted tachyon lines are dashed and the solid line is any particular twisted sector of the orbifold.

$$\hat{A}_n^{(\lambda)}(\{T\})_I = S^T \frac{n \dots 1}{2\pi\lambda^2} S = \hat{A}_n^{(\lambda)}(\{T\})$$

Fig. 1: The twisted trees of type I are independent of I.

There are other useful forms of the twisted trees. For example, we may use the integral representations in Eq. (2.6) and the boost relation in Eq. (2.8) to put the four-point amplitudes in the form

$$\begin{aligned} \hat{A}_4^{(\lambda)}(\{\mathcal{T}\}) &= \int \frac{d^2 z}{2\pi\lambda^2} \sum_{r=0}^{\lambda-1} \times \\ &\times S^T \langle -T^{(4)} | R(\hat{g}(\mathcal{T}^{(3)}, 1, 1) \hat{g}(\mathcal{T}^{(2)}, \bar{z}e^{-2\pi ir}, ze^{2\pi ir}) | T^{(1)} \rangle S \end{aligned} \quad (2.13)$$

where R is radial ordering and z is now integrated over the full complex plane. Note that the integrands here are explicitly monodromy-invariant ($z \rightarrow ze^{2\pi i}$) as they should be. This formula can be straightforwardly generalized to all the n -point amplitudes, and each of these can be further simplified by the substitution $\hat{g} \rightarrow \hat{g}$, $S \rightarrow 1$ as above.

2.3 Extended Ward Identities and Physical States

Our next topic is the physical spectrum of the twisted orbifold-string sectors, as implied by the twisted trees.

For each twisted sector σ , I define the extended (left-mover) gauge operators

$$\hat{W}_r \left(m + \frac{r}{\lambda} \right) \equiv t_{-r} \hat{L}_r \left(m + \frac{r}{\lambda} \right) - \left(\hat{L}_0(m) + \left(m + \frac{r}{\lambda} \right) - \hat{a}_\lambda \right) \quad (2.14a)$$

$$\bar{r} = 0, \dots, \lambda - 1 \quad (2.14b)$$

and right-mover copies with $\hat{L} \rightarrow \hat{\bar{L}}$. Useful identities for the study of these gauges include:

$$[t_{-r} \hat{L}_r \left(m + \frac{r}{\lambda} \right) - \hat{L}_0(m), \hat{g}(\mathcal{T}^{(i)}, 1, 1)] = \left(m + \frac{r}{\lambda} \right) \hat{g}(\mathcal{T}^{(i)}, 1, 1) \quad (2.15a)$$

$$\hat{W}_r \left(m + \frac{r}{\lambda} \right) \hat{g}(\mathcal{T}^{(i)}, 1, 1) = \hat{g}(\mathcal{T}^{(i)}, 1, 1) \left(t_{-r} \hat{L}_r \left(m + \frac{r}{\lambda} \right) - \hat{L}_0(0) + \hat{a}_\lambda \right) \quad (2.15b)$$

$$\begin{aligned} & \left(t_{-r} \hat{L}_r \left(m + \frac{r}{\lambda} \right) - \hat{L}_0(0) + \hat{a}_\lambda \right) \hat{D}[\hat{L}_0(0), \hat{\bar{L}}_0(0)] \\ & \simeq \hat{D}[\hat{L}_0(0) + m + \frac{r}{\lambda}, \hat{\bar{L}}_0(0)] \hat{W}_r \left(m + \frac{r}{\lambda} \right). \end{aligned} \quad (2.15c)$$

Here Eq. (2.15a) is an orbifold generalization of the so-called "stability condition" of Refs. [1,31], and Eq. (2.15b) follows from (2.15a). The proof of Eq. (2.15c) requires the extended Virasoro algebra (2.2d), and the symbol \simeq means that I have also used the "cancelled propagator" argument (see e.g. Ref. [12]).

Then we see that the extended gauges are operative in the twisted trees³

$$\hat{W}_r \left(\left(m + \frac{r}{\lambda} \right) > 0 \right) \hat{g}(\mathcal{T}^{(m)}, 1, 1) \hat{D} \dots \hat{D} \hat{g}(\mathcal{T}^{(1)}, 1, 1) |\chi\rangle S = 0 \quad (2.16)$$

so long as the physical oscillator states $\{|\chi\rangle\}$ satisfy the *extended physical state condition*:

$$\left(\hat{L}_r \left(\left(m + \frac{r}{\lambda} \right) \geq 0 \right) - \hat{a}_\lambda \delta_{m+\frac{r}{\lambda}, 0} \right) |\chi\rangle = 0, \quad \bar{r} = 0, \dots, \lambda - 1. \quad (2.17)$$

This condition (and a right-mover copy on the same states $\{|\chi\rangle\}$) is the operator realization of the extended classical Virasoro constraints of \mathbb{Z}_λ -twisted permutation gravity [28].

³Although there is no need to do so here, we know from Subsec. 2.2 that the replacement $t_r \rightarrow 1_\lambda$, $\hat{g} \rightarrow \hat{\bar{g}}$ can be made in all the operators of Eq. (2.16) on S .

Of central importance, the case $\lambda = 2$ of the extended physical state condition (2.17) is in precise agreement with that obtained from the twisted BRST system of $\hat{c} = 52$ matter in Ref. [29]. Moreover, exactly this form of the extended physical state condition was conjectured for $\lambda \geq 3$ in Ref. [27]. Note finally that the ground state $|T\rangle$ of the twisted sector in Eq. (2.7b) is indeed a physical state for each λ .

What is perhaps surprising is that, *as a string theory*, the physical spectrum of each twisted $\hat{c} = 26\lambda$ sector is in fact the same as that of an ordinary untwisted string at $c = 26$! This *spectral equivalence* was discussed in detail for the \mathbb{Z}_2 orbifold-string theory in Ref. [30], using the (inverse of the) order-two orbifold-induction procedure [10]. To see the spectral equivalence here for all prime λ one need only generalize the discussion of Ref. [30], using now the (inverse of the) order- λ *orbifold induction procedure* [10]. As seen explicitly in the following set of equations, the general procedure maps the twisted matter of the extended Virasoro generators (2.2) and the extended physical state condition (2.17) to the (unhatted) equivalent conventional untwisted system at $c = 26$:

$$L(\lambda m + r) \equiv \lambda \hat{L}_r \left(m + \frac{r}{\lambda} \right) - \frac{13}{12}(\lambda^2 - 1)\delta_{m+\frac{r}{\lambda},0} \quad (2.18a)$$

$$J_\mu(\lambda m + r) \equiv J_{r\mu} \left(m + \frac{r}{\lambda} \right) \quad (2.18b)$$

$$L(M) = -\frac{\eta^{\mu\nu}}{2} \sum_{P \in \mathbb{Z}} \circ J_\mu(P) J_\nu(M - P) \circ_M, \quad M \in \mathbb{Z} \quad (2.18c)$$

$$[J_\mu(M), J_\nu(N)] = -\eta_{\mu\nu} M \delta_{M+N,0} \quad (2.18d)$$

$$[L(M), J_\mu(N)] = -N J_\mu(M + N) \quad (2.18e)$$

$$[L(M), L(N)] = (M - N)L(M + N) + \frac{26}{12}M(M^2 - 1)\delta_{M+N,0} \quad (2.18f)$$

$$(L(M \geq 0) - \delta_{M,0}) |\chi\rangle = 0. \quad (2.18g)$$

A right-mover copy of this system is similarly obtained, and the conventional physical states $\{|\chi\rangle\}$ in Eq. (2.18g) are exactly the *same* physical states, with the same physical spectrum, as those defined by the extended physical state condition (2.17) – each state $|\chi\rangle$ having now been reexpressed via Eq. (2.18b) in terms of the untwisted modes $\{J, \bar{J}\}$. As emphasized for the case $\lambda = 2$ in Ref. [27], this spectral equivalence holds for the $\hat{c} = 26\lambda$ and $c = 26$ systems *only* when they are considered as string theories (truncated respectively by the physical state conditions (2.17) and/or (2.18g)), and *not* for the CFT's themselves. We shall revisit this equivalence from other viewpoints in the next subsection and Sec. 3.

2.4 Example: Four-Point Amplitude in $U(1)^{52}/\mathbb{Z}_2$

As an explicit example at $\lambda = 2$, I will evaluate the twisted four-point string amplitude (2.9) for the single twisted sector of the permutation orbifold $(U(1)^{26} \times U(1)^{26})/\mathbb{Z}_2$ at $\hat{c} = 52$. Towards this, we need the explicit form of the twisted vertex operators [19] for this case:

$$\hat{g}(\mathcal{T}, \bar{z}, z) = \hat{g}_-(\mathcal{T}, \bar{z}) \hat{g}_+(\mathcal{T}, z) \quad (2.19a)$$

$$\begin{aligned} \hat{g}_+(\mathcal{T}, z) &= z^{-\frac{1}{2}} e^{-i\tau_0 T_\mu \hat{q}^\mu} e^{-\frac{1}{2}\tau_0 T \cdot \hat{J}_0(0)} \times \\ &\times \exp\left(\frac{1}{2}\tau_0 T \cdot \sum_{m \neq 0} \hat{J}_0(m) \frac{z^{-m}}{m}\right) \times \end{aligned} \quad (2.19b)$$

$$\begin{aligned} &\times \exp\left(\frac{1}{2}\tau_1 T \cdot \sum_m \hat{J}_1\left(m + \frac{1}{2}\right) \frac{z^{-(m+\frac{1}{2})}}{m + \frac{1}{2}}\right) \\ &\left[\hat{q}^\mu, \hat{J}_{r\nu}\left(m + \frac{r}{\lambda}\right)\right] = i\delta_\nu^\mu \delta_{m+\frac{r}{\lambda}, 0}, \quad [\hat{q}^\mu, \hat{q}^\nu] = 0. \end{aligned} \quad (2.19c)$$

Here I have used $t_0 = \tau_0 = \mathbb{1}_2$ and $t_1 = \tau_1$ (the first Pauli matrix), and dot products are defined with the target-space metric η in Eq. (2.2e). The form of the right-mover \hat{g}_- is the same as \hat{g}_+ with $z \rightarrow \bar{z}$, $\hat{J} \rightarrow \hat{\bar{J}}$ and $\hat{q} \rightarrow \hat{\bar{q}}$. Then it is straightforward to compute the orbifold correlator

$$\begin{aligned} &\langle -T^{(4)} | R(\hat{g}(\mathcal{T}^{(3)}, \bar{z}_3, z_3) \hat{g}(\mathcal{T}^{(2)}, \bar{z}_2, z_2)) | T^{(1)} \rangle \\ &= \mathbb{1}_2 \delta^{26} \left(\sum_{i=1}^4 T^{(i)} \right) |z_3|^{-(T^{(3)} \cdot T^{(2)} + 1)} |z_2|^{-(T^{(2)} \cdot T^{(1)} + 1)} \times \\ &\times |z_3 - z_2|^{-T^{(3)} \times T^{(2)}} \left| \frac{\sqrt{z_3} - \sqrt{z_2}}{\sqrt{z_3} + \sqrt{z_2}} \right|^{-T^{(3)} \cdot T^{(2)}} \end{aligned} \quad (2.20)$$

which is proportional to the unit matrix $\mathbb{1}_2$ because only even powers of τ_1 survive the contraction.

The corresponding four-point orbifold-string amplitude

$$\begin{aligned} \hat{A}_4^{(2)}(\{\mathcal{T}\}) &= \delta^{26} \left(\sum_{i=1}^4 T^{(i)} \right) S^T \mathbb{1}_2 S \times \\ &\times \int \frac{d^2 z}{8\pi} |z|^{-(T^{(2)} \cdot T^{(1)} + 1)} |1 - z|^{-T^{(2)} \cdot T^{(3)}} \times \\ &\times \left(\left| \frac{1 - \sqrt{z}}{1 + \sqrt{z}} \right|^{-T^{(2)} \cdot T^{(3)}} + \left| \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right|^{-T^{(2)} \cdot T^{(3)}} \right) \end{aligned} \quad (2.21)$$

is then obtained from Eq. (2.13) at $\lambda = 2$. According to Eq. (2.10), the spinor factor $S^T \mathbb{1}_2 S$ here is trivial – and moreover the identities

$$|1 - z|^a \left| \frac{1 \mp \sqrt{z}}{1 \pm \sqrt{z}} \right|^a = |1 \mp \sqrt{z}|^{2a} \quad (2.22)$$

can be used to further simplify this result. The square roots in these equations reflect the half-integer moding of the extended Virasoro generators (2.2a) and twisted vertex operators (2.19) of the twisted sector in this case.

In fact, all the roots in the expression (2.21) can be removed by the following change of variable

$$z = u^2, \quad d^2 z = 4|u|^2 d^2 u \quad (2.23a)$$

$$\int_0^\pi d^2 u |u|^b (|1 - u|^a + |1 + u|^a) = \int_{-\pi}^\pi d^2 u |u|^b |1 - u|^a \quad (2.23b)$$

where the range of $\arg(u)$ is shown on the integrals. This brings us to our final result:

$$\hat{A}_4^{(2)}(\{\mathcal{T}\}) = \hat{A}_4^{(2)}(\{T\})_I, \quad I = 0, 1 \quad (2.24a)$$

$$= \delta^{26} \left(\sum_{i=1}^4 T^{(i)} \right) \int_{-\pi}^\pi \frac{d^2 u}{2\pi} |u|^{-T^{(2)} \cdot T^{(1)}} |1 - u|^{-2T^{(2)} \cdot T^{(3)}}. \quad (2.24b)$$

If we identify the dimensionful momenta $\{k\}$ and the closed-string Regge slope α'_c as

$$T = \sqrt{\alpha'_c} k, \quad (T^{(i)})^2 = \alpha'_c (k^{(i)})^2 = -2 \quad (2.25)$$

then the integral in Eq. (2.24b) is recognized as the Virasoro-Shapiro amplitude [10,12] of the ordinary critical unoriented closed string.

Our interpretation of the result (2.24) is shown as the $\lambda = 2$ case of Fig. 2 (see Subsec. 2.2 and Fig. 1): For each type $\{I = 0, 1\}$ of external untwisted tachyon, the cross-channel of the I -independent amplitude $\hat{A}_{4I}^{(2)}$ shows the spectrum of an ordinary untwisted closed string of type I . This duality is perhaps not surprising since we already knew that a) the external untwisted tachyons had the same mass as the the ground-state tachyon of the twisted sector, b) the physical spectrum of the twisted sector is identical to that of an ordinary closed string, and c) the twisted trees in Eq. (2.12b) are symmetrized with respect to the vertex operators \hat{g}_I . I emphasize that the external tachyons of type I couple only to the internal untwisted closed string of type I – which is consistent with the fact that (before symmetrization) the untwisted sector $U(1)^{52} = (U(1)^{26} \times U(1)^{26})$ of the orbifold is nothing but two *decoupled* copies of the untwisted closed string.

$$S^r \text{---} S = S^r \text{---} S, \quad I = 0, 1, \dots, \lambda-1$$

Fig. 2: The cross channel is the untwisted sector.

The principles of the previous paragraph hold for emission of an arbitrary number of untwisted tachyons of type I , and it is straightforward to check that the I -independent twisted trees $\{\hat{A}_{nI}^{(2)}\}$ are also proportional to ordinary Virasoro-Shapiro n -point amplitudes. For brevity I will not give these computations here but, as illustrated in our four-point example above, the steps are nothing but a complexified version of those given in the n -point twisted open-string computation of Ref. [31]. Given the known decoupling of zero-norm states in the Virasoro-Shapiro trees, this provides the expected [28] no-ghost theorem for the twisted sector of the \mathbb{Z}_2 permutation-orbifold string. Indeed, except that $I = 0, \dots, \lambda - 1$, the principles of the previous paragraph hold for the orbifolds with all prime λ , so I expect the *same* Virasoro-Shapiro amplitudes $\hat{A}_{nI}^{(\lambda)} \sim \hat{A}_{nI}^{(2)}$ for $\lambda \geq 3$. This expectation should be checked directly from the twisted trees (2.9), using the explicit forms of the (higher-fractional-moded) vertex operators given in Ref. [19].

3 The Twisted Sectors on the Torus

3.1 The Cosmological Constant Λ

To establish a language for the development below, I begin with a short *review* of the cosmological constant (see e.g. Ref. [12]).

In D -dimensional Euclidean space, the *naive* or *particle-theoretic* form of the one-loop cosmological constant (ground state energy/volume) for any relativistic theory is

$$\Lambda_N = \frac{E_0}{V} = \sum_j \int (d^{D-1}k) \frac{1}{2} \omega_j(\vec{k}) = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \int (d^D k) \sum_j e^{-t(k^2 + m_j^2)} \quad (3.1a)$$

$$(d^D k) = \frac{d^D k}{(2\pi)^D}, \quad \omega_j(\vec{k}) = \sqrt{\vec{k}^2 + m_j^2}, \quad k^2 = k_0^2 + \vec{k}^2 \quad (3.1b)$$

where the sum is over all *particles* $\{j\}$ in the theory. For the ordinary untwisted closed string in $D = 26$ Euclidean dimensions, we recall that

$$L(0) = \frac{J^2(0)}{2} + R, \quad \bar{L}(0) = \frac{\bar{J}^2(0)}{2} + \bar{R}, \quad J(0) = \sqrt{\alpha'_c} k \quad (3.2a)$$

$$D = \frac{\delta_{L(0), \bar{L}(0)}}{L(0) + \bar{L}(0) - 2}, \quad \delta_{L(0), \bar{L}(0)} = \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i\phi(L(0) - \bar{L}(0))} \quad (3.2b)$$

where operator D is the string propagator. Then Eq. (3.1) gives the naive or particle-theoretic form of the string cosmological constant

$$\begin{aligned} (\alpha'_c)^{13} \Lambda_N &= -\frac{1}{2} \int_0^\infty \frac{dt}{t} \int (d^{26} J(0)) \text{Tr} \left(e^{-t(L(0) + \bar{L}(0) - 2)} \delta_{L(0), \bar{L}(0)} \right) \\ &= -\frac{1}{2} \int_{F_N} \frac{d^2 \tau}{(\text{Im } \tau)^2} Z(\bar{\tau}, \tau), \quad \tau \equiv \frac{1}{2\pi}(\phi + it) \end{aligned} \quad (3.3a)$$

$$\begin{aligned} Z(\tau, \bar{\tau}) &= \text{Im } \tau \int (d^{26} J(0)) \text{Tr} \left(e^{2\pi i \tau (L(0) - 1)} e^{-2\pi i \bar{\tau} (\bar{L}(0) - 1)} \right) \\ &= \frac{1}{(8\pi^2)^{13}} \frac{1}{(\text{Im } \tau)^{12}} |\eta(e^{2\pi i \tau})|^{-48} \end{aligned} \quad (3.3b)$$

$$F_N : \quad |\text{Re } \tau| \leq \frac{1}{2}, \quad \text{Im } \tau \geq 0 \quad (3.3c)$$

which is integrated over the *naive* or *particle-theoretic* region F_N of moduli space. Here $Z(\tau, \bar{\tau})$ is the modular-invariant $c = 26$ closed-string partition function, $\eta(e^{2\pi i\tau})$ is the Dedekind η -function, and I have assumed that the traces in Eq. (3.3) sum only over the physical states. The summation over particle-theoretic contributions described here is an example of the naive, operator sewing [34,12] of any string loop, and the same naive region F_N is encountered as well in the naive sewing of closed-string loops with insertions.

Dividing by $SL(2, \mathbb{Z})$, we then obtain the *physical* or *string-theoretic* cosmological constant Λ

$$F_N \rightarrow F : |\operatorname{Re} \tau| \leq \frac{1}{2}, \quad \operatorname{Im} \tau \geq 0, \quad |\tau| \geq 1 \quad (3.4a)$$

$$(\alpha'_c)^{13} \Lambda_N \rightarrow (\alpha'_c)^{13} \Lambda = -\frac{1}{2} \int_F \frac{d^2 \tau}{(\operatorname{Im} \tau)^2} Z(\tau, \bar{\tau}) \quad (3.4b)$$

of the untwisted closed string, where F is the standard fundamental region. If so desired, we may formally return to the naive, particle-theoretic form ($F \rightarrow F_N, \Lambda \rightarrow \Lambda_N$) by ignoring the last (circular arc) constraint in Eq. (3.4a) – and indeed this return is necessary to see the particle content (3.1) of the theory.

3.2 The Twisted Contribution to $\hat{\Lambda}$

In this subsection I compute the contribution of the twisted sectors of the permutation orbifold $U(1)^{26\lambda}/\mathbb{Z}_\lambda$ to the orbifold cosmological constant $\hat{\Lambda}$.

For each twisted sector, we will need the following identities:

$$\hat{L}_0(0) = \frac{\hat{J}^2(0)}{2\lambda} + \hat{R}, \quad \hat{\bar{L}}_0(0) = \frac{\hat{\bar{J}}^2(0)}{2\lambda} + \hat{\bar{R}} \quad (3.5a)$$

$$\delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0)} = \delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0) \bmod 1} \int_{-\pi}^{\pi} \frac{d\phi}{2\pi} e^{i\phi(\hat{L}_0(0) - \hat{\bar{L}}_0(0))} \quad (3.5b)$$

$$\delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0) \bmod 1} = \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} e^{2\pi i r (\hat{L}_0(0) - \hat{\bar{L}}_0(0))}. \quad (3.5c)$$

Recall that the ordinary Kronecker delta on the left of Eq. (3.5b) is also the numerator of the twisted propagator \hat{D} in Eq. (2.5), and indeed the identities (3.5b,c) were used to obtain the integral representations of \hat{D} in Eq. (2.6).

Then Eq. (3.1) gives the naive or particle-theoretic contribution $\hat{\hat{\Lambda}}_N$ of all $\lambda - 1$ twisted sectors to the orbifold cosmological constant

$$\begin{aligned} (\alpha'_c)^{13} \hat{\hat{\Lambda}}_N^{(\lambda)} &= -\frac{(\lambda-1)}{2} \int_0^\infty \frac{dt}{t} \int (d^{26} \hat{J}(0)) Tr \left(e^{-t(\hat{L}_0(0) + \hat{\bar{L}}_0(0) - 2\hat{a}_\lambda)} \delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0)} \right) \\ &= -\frac{(\lambda-1)}{2} \int_{F_N} \frac{d^2 \tau}{(\text{Im } \tau)^2} \hat{Z}_\lambda(\tau, \bar{\tau}) \end{aligned} \quad (3.6a)$$

$$\hat{Z}_\lambda(\tau, \bar{\tau}) \equiv \text{Im } \tau \int (d^{26} \hat{J}(0)) Tr \left(e^{2\pi i \tau (\hat{L}_0(0) - \hat{a}_\lambda)} e^{-2\pi i \bar{\tau} (\hat{\bar{L}}_0(0) - \hat{a}_\lambda)} \delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0) \bmod 1} \right) \quad (3.6b)$$

where $\hat{Z}_\lambda(\tau, \bar{\tau})$ will be called the *twisted partition function*. I will also define a physical or string-theoretic version $\hat{\hat{\Lambda}}$ of these contributions by the formal substitution $F_N \rightarrow F$

$$(\alpha'_c)^{13} \hat{\hat{\Lambda}}^{(\lambda)} \equiv -\frac{(\lambda-1)}{2} \int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} \hat{Z}_\lambda(\tau, \bar{\tau}) \quad (3.7)$$

although strictly speaking there is no need to divide by $SL(2, \mathbb{Z})$ until the modular-invariant completion of $\hat{Z}_\lambda(\tau, \bar{\tau})$ in Sec. 4.

The twisted partition function can be simplified by the order- λ orbifold induction procedure in Eq.(2.18), which implies the following relations:

$$\hat{L}_0(0) - \hat{a}_\lambda = \frac{1}{\lambda} (L_0(0) - 1), \quad \hat{\bar{L}}_0(0) - \hat{a}_\lambda = \frac{1}{\lambda} (\bar{L}_0(0) - 1), \quad \hat{J}(0) = J(0) \quad (3.8a)$$

$$\delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0) \bmod 1} = \frac{1}{\lambda} \sum_{r=0}^{\lambda-1} e^{2\pi i \frac{r}{\lambda} (L_0(0) - \bar{L}_0(0))} = \delta_{L_0(0), \bar{L}_0(0) \bmod \lambda} \quad (3.8b)$$

$$\hat{D} = \frac{\delta_{\hat{L}_0(0), \hat{\bar{L}}_0(0)}}{\lambda(\hat{L}_0(0) + \hat{\bar{L}}_0(0) - 2\hat{a}_\lambda)} = \frac{\delta_{L_0(0), \bar{L}_0(0)}}{L_0(0) + \bar{L}_0(0) - 2} = D. \quad (3.8c)$$

I have included here the closely-related propagator identity (3.8c), which was in fact used to fix the scale of the twisted propagator \hat{D} in Subsec. (2.2). Then we find from Eqs. (3.6b) and (3.8) that

$$\begin{aligned}
\hat{Z}_\lambda(\tau, \bar{\tau}) &= \text{Im } \tau \int (d^{26} J(0)) \text{Tr} \left(e^{2\pi i \frac{\tau}{\lambda} (L(0)-1)} e^{-2\pi i \frac{\bar{\tau}}{\lambda} (\bar{L}(0)-1)} \delta_{L(0), \bar{L}(0) \bmod \lambda} \right) \\
&= \text{Im} \left(\frac{\tau}{\lambda} \right) \sum_{r=0}^{\lambda-1} \int (d^{26} J(0)) \text{Tr} \left(e^{2\pi i \frac{\tau+r}{\lambda} (L(0)-1)} e^{-2\pi i \frac{\bar{\tau}+r}{\lambda} (\bar{L}(0)-1)} \right) \\
&= \sum_{r=0}^{\lambda-1} Z \left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda} \right)
\end{aligned} \tag{3.9}$$

where $Z(\tau, \bar{\tau})$ is the $c = 26$ string partition function in Eq. (3.3b). In passing from Eq. (3.6b) to Eq. (3.9) I have used the fact that the physical Hilbert spaces of the $\hat{c} = 26\lambda$ and the $c = 26$ descriptions are isomorphic (see Refs. [13] and [30]) – and hence the traces are identical. Up to a numerical factor of λ (associated to the “extra” prefactor $\text{Im } \tau$ in Eqs. (3.3b) and (3.6b)), the last line of the result (3.9) is the standard form [8] of the twisted partition function for the $\hat{c} = 26\lambda$ orbifold conformal field theories.

3.3 Spectral Equivalence in the Cosmological Constant

We saw in Eq. (3.9) that the partition function \hat{Z}_λ of each twisted $\hat{c} = 26\lambda$ string is proportional to the standard twisted partition function of the corresponding permutation-orbifold CFT. This is natural enough, except that the spectra of the permutation-orbifold CFT’s are notoriously intricate [7,10] – while we know from Ref. [30] and Sec. 2 that, *as a string theory* restricted by the extended physical state condition (2.17), the physical spectrum of each twisted sector is equivalent to that of an ordinary untwisted $c = 26$ closed string. The question is then how to see this spectral equivalence at the level of the orbifold cosmological constant, and the answer must lie in the final τ -integration of Eqs. (3.6) and (3.7). Indeed the following results on the torus stand in close analogy with those given above (see Subsecs. 2.3 and 2.4) for the twisted trees – where the spectral equivalence is seen only after the z -integrations over the CFT correlators on the sphere.

Spectral equivalence on the torus can be understood in terms of two lemmas, whose proof will be sketched after the statement of the lemmas.

The first lemma is a formal integration identity

$$\int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) = \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) \quad (3.10a)$$

$$\rightarrow \hat{\Lambda}_N^{(\lambda)} = (\lambda - 1)\Lambda_N \quad (3.10b)$$

where F_N is the naive region in Eq. (3.3c). This identity tells us that, as required by spectral equivalence on the sphere, the integrated particle-theoretic contribution of each twisted $\hat{c} = 26\lambda$ sector is the same as the integrated particle-theoretic contribution Λ_N (see Eq. (3.31)) of an ordinary closed string.

The second lemma is a physical or string-theoretic version of the same spectral equivalence

$$\int_F \frac{d^2\tau}{(\text{Im } \tau)^2} \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) = \int_{\hat{F}_\lambda} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) \quad (3.11a)$$

$$\rightarrow \hat{\Lambda}^{(\lambda)} = -\frac{(\lambda - 1)}{2} \int_{\hat{F}_\lambda} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) \neq (\lambda - 1)\Lambda \quad (3.11b)$$

$$\hat{F}_\lambda : |\text{Re } \tau| \leq \frac{1}{2}, \quad \text{Im } \tau \geq 0, \quad |\lambda\tau + r| \geq 1, \quad r = 0, \pm 1, \dots, \pm \left\lfloor \frac{\lambda}{2} \right\rfloor \quad (3.11c)$$

where F is the fundamental region and $\lfloor x \rfloor$ is the floor of x . This remarkable relation reconfirms at the physical level that the spectrum of each twisted sector of $U(1)^{26\lambda}/\mathbb{Z}_\lambda$ is that of an ordinary string, but in this description the contribution of each sector must be cut off by the *new modular region* \hat{F}_λ .

For reference, Fig. 3 shows the first few modular regions $\{\hat{F}_\lambda\}$ as the areas above the circular arcs.

In fact, both of the lemmas in Eqs. (3.10a) and (3.11a) are nothing but “cut and paste” identities which hold for any function $Z(\tau, \bar{\tau})$ which is periodic with period one:

$$Z(\tau' + 1, \bar{\tau}' + 1) = Z(\tau', \bar{\tau}'), \quad \tau' = \frac{\tau}{\lambda} \quad (3.12a)$$

$$\rightarrow \hat{Z}_\lambda(\tau + 1, \bar{\tau} + 1) = \hat{Z}_\lambda(\tau, \bar{\tau}). \quad (3.12b)$$

This includes any modular-invariant $Z(\tau, \bar{\tau})$, such as the partition function (3.3b) of the ordinary closed string. The proofs of these identities are straightforward but can appear somewhat involved at the algebraic level, as seen for

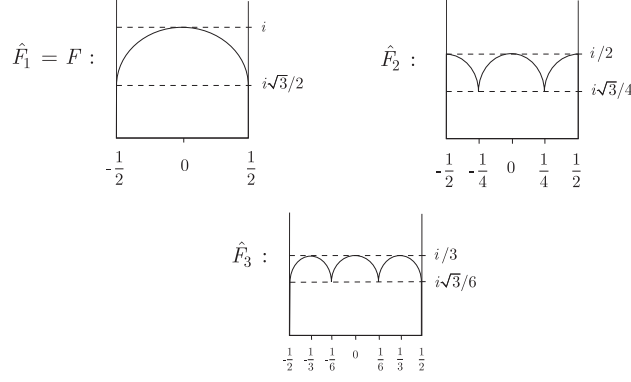


Fig. 3: The first three regions \hat{F}_λ

example in the following sketch for the case $\lambda = 2$ of the identity in Eq. (3.11a). Begin with the integration over the two terms

$$\hat{Z}_2(\tau, \bar{\tau}) = Z\left(\frac{\tau}{2}, \frac{\bar{\tau}}{2}\right) + Z\left(\frac{\tau+1}{2}, \frac{\bar{\tau}+1}{2}\right) \quad (3.13)$$

on the left side of the identity, and follow the changes of variable:

$$\text{a) } \tau' = \begin{cases} \tau + 1 & , \quad -\frac{\pi}{2} < \text{Re } \tau < 0 \\ \tau & , \quad 0 < \text{Re } \tau < \frac{\pi}{2} \end{cases} \quad (\text{periodicity of } \hat{Z}_2) \quad (3.14a)$$

$$\text{b) } \tau'' = \frac{\tau'}{2} \quad (\text{rescale}) \quad (3.14b)$$

$$\text{c) } \tau''' = \tau'' - \frac{1}{2} \quad (\text{second term of (3.13) only}). \quad (3.14c)$$

After step (a) for example, the integration region is $0 \leq \text{Re } \tau' \leq 1$ with a cusp at $\frac{1}{2}$. Both integrands are equal to $Z(\tau, \bar{\tau})$ after the shift (c), and the integration region pastes to the modular region \hat{F}_2 .

The proof of the particle-theoretic counterpart in Eq. (3.10a) follows the same steps, now ignoring the circular arcs of the modular regions $\{\hat{F}_\lambda\}$. In this case of course, the singularity of the integrands at $\text{Im } \tau = 0$ means that the steps and therefore the result are formal.

4 About the Untwisted Contribution to $\hat{\Lambda}$

The contribution of the untwisted, permutation-symmetric sector of the orbifold-string system is surprisingly more difficult to understand.

4.1 Trouble with the Standard CFT Prescription

Given λ copies of any conformal field theory with discrete spectrum and partition function $Z(\tau, \bar{\tau})$, the standard CFT prescription for the full modular-invariant partition function of the cyclic permutation orbifold was given in Ref. [8]. After integration over the fundamental region, this prescription gives the following model $\hat{\Lambda}_G$ of the full orbifold cosmological constant:

$$(\alpha'_c)^{13} \hat{\Lambda}_G = -\frac{\lambda}{2} \int_F \frac{d^2\tau}{(\text{Im } \tau)^2} \left\{ \frac{Z^\lambda(\tau, \bar{\tau})}{\lambda} + \frac{\lambda-1}{\lambda} \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \right\} \quad (4.1)$$

From our point of view, the overall scale of this model is set by its final sum of terms, which correctly integrates to the twisted contribution $\hat{\Lambda}$ in Eqs. (3.7) and (3.11). The other two terms in Eq. (4.1) are contributions from the untwisted sector. In particular, the second term provides the modular-invariant completion of the twisted contribution, the sum of all terms in the square brackets being a Hecke transform of order λ . From the point of view of Ref. [8], the first term is the original unsymmetrized partition function $\prod_I Z_I = Z^\lambda$ of the untwisted sector, and the second term is a correction to the first term for the symmetrized counting – using the discreteness of the spectrum – of the so-called “diagonal” states $\{|x\rangle \otimes \dots \otimes |x\rangle\}$ of the untwisted sector. In our discussion above we have assumed for simplicity that the spectrum of the ordinary closed-string partition function $Z(\tau, \bar{\tau})$ is not discrete (hence the extra powers of $\text{Im } \tau$ in Eq. (3.3b)), but this mismatch can be remedied by compactification.

There is of course no objection to the partition function of Ref. [8] as an orbifold CFT, and indeed I had expected the form (4.1) to provide the desired full cosmological constant of the orbifold – but in fact there is a physical problem with the untwisted contribution to this model *as a string system*. The point is that the ordinary closed-string partition function $Z(\tau, \bar{\tau})$

in Eq. (3.3b) involves a single momentum integration

$$Z = \int (d^{26}J(0)) \dots = \int (d^{26}k) \dots \quad (4.2)$$

and this gives an untwisted contribution to the model which is proportional to λ *momentum integrations* because

$$Z^\lambda = \int \left(\prod_{I=0}^{\lambda-1} (d^{26}k_I) \right) \dots \quad (4.3)$$

But this *higher-loop term* in $\hat{\Lambda}_G$ contradicts the standard one-loop particle-theoretic form (3.1) of any cosmological constant!

On the basis of this simple observation, we are forced to conclude for all prime $\lambda \geq 1$ that the untwisted contribution (and hence all the contributions) to the orbifold cosmological constants of $U(1)^{26\lambda}/\mathbb{Z}_\lambda$ must be *linear* in the partition function $Z(\tau, \bar{\tau})$ of the ordinary closed string. Although the constructions of this paper will be complete only for these particular orbifolds, I emphasize that the same conclusion (linear in $\int (d^{26}k)$) applies as well to the cosmological constant of the general permutation orbifold in Eq. (1.1).

4.2 Particle-Theoretic Form of the Untwisted Contribution

We can further support the conclusion of the previous subsection by considering the untwisted contribution to the cosmological constant in an analogous set of permutation-invariant quantum field theories. Consider in particular the \mathbb{Z}_λ -invariant Hamiltonian

$$H = \int d^{25}x \left[\sum_j \sum_{r=0}^{\lambda-1} \frac{1}{2} (\pi_{jr}^2 + |\nabla \phi_{jr}|^2 + m_j^2 \phi_{jr}^2) \right] + V_\lambda[\phi] \quad (4.4)$$

where $V_\lambda[\phi]$ is any interaction which is invariant under cyclic permutations $\{\phi_{jr} \rightarrow \phi_{j,r+1}\}$ of the copies $\{r\}$ of each particle j . At each value of λ , the one-loop contribution Λ_λ to the exact cosmological constant

$$\Lambda_{\text{exact}} = \langle 0 | (H/V) | 0 \rangle = \Lambda_\lambda + (\text{higher loops}) \quad (4.5a)$$

$$\Lambda_\lambda = \sum_j \sum_{r=0}^{\lambda-1} \int (d^{25}k) \frac{1}{2} \sqrt{\vec{k}^2 + m_j^2} \quad (4.5b)$$

is independent of the interaction and *automatically symmetric* under \mathbb{Z}_λ . Then we find that

$$\Lambda_\lambda = \lambda \Lambda \quad (4.6a)$$

$$\Lambda = -\frac{1}{2} \int_0^\infty \frac{dt}{t} \int (d^{26}k) \sum_j e^{-t(k^2 + m_j^2)} \quad (4.6b)$$

where Λ is the one-loop cosmological constant for a single copy of all the particles $\{j\}$ in the theory.

Promoting Eq. (4.6) to the string theory (see Subsec. 2.1) of λ copies of $U(1)^{26}$, we learn that the untwisted particle-theoretic contribution to the orbifold cosmological constant is indeed *linear* in the ordinary closed-string partition function

$$(\alpha'_c)^{13} \hat{\Lambda}_N^{(\text{sym})} = -\frac{\lambda}{2} \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) \quad (4.7)$$

where $Z(\tau, \bar{\tau})$ and the naive region F_N are given Eq. (3.3). At this level then, the one-loop contribution $\hat{\Lambda}_N^{(\text{sym})} = \lambda \Lambda_N$ of the untwisted sector is equivalent to that of λ ordinary closed strings, and this is consistent with the cross-channel behaviour of the twisted trees (see Secs. 2 and 6). The reasoning above is easily extended to see that $\hat{\Lambda}_N^{(\text{sym})} = K \Lambda_N$ for the untwisted particle-theoretic contribution to the general permutation orbifold $U(1)^{26K}/H(\text{perm})$ on K copies.

The particle-theoretic result in Eq. (4.7) is in fact a natural structure in the permutation-orbifold conformal field theory [8,13]. In particular, the following formal integration identity

$$\lambda \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) = \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\lambda\tau, \lambda\bar{\tau}). \quad (4.8)$$

shows that, at this naive level, summing over λ ordinary strings is equivalent to summing over the so-called *diagonal states* $\{|x\rangle \otimes \dots \otimes |x\rangle\}$ of the untwisted sector of the orbifold. The identity (4.8) provides yet another *spectral equivalence*, which parallels the role of Eq. (3.10a) in the twisted sector. The identity itself can be proven directly from the explicit form of $Z(\tau, \bar{\tau})$ in Eq. (3.3b), or by the following formal steps

$$\int_{-1/2}^{1/2} d(\operatorname{Re} \tau) e^{2\pi i \lambda \operatorname{Re} \tau (L(0) - \bar{L}(0))} = \delta_{L(0), \bar{L}(0)} \quad (4.9a)$$

$$\int_{F_N} \frac{d^2 \tau}{(\operatorname{Im} \tau)^2} Z(\lambda \tau, \lambda \bar{\tau}) = \int_0^\infty \frac{d(\operatorname{Im} \tau)}{(\operatorname{Im} \tau)^2} f(\lambda \operatorname{Im} \tau) = \lambda \int_0^\infty \frac{d(\operatorname{Im} \tau)}{(\operatorname{Im} \tau)^2} f(\operatorname{Im} \tau) \quad (4.9b)$$

$$f(\operatorname{Im} \tau) \equiv \operatorname{Im} \tau \int (d^{26} J(0)) \operatorname{Tr} \left(e^{-2\pi \operatorname{Im} \tau (L(0) + \bar{L}(0) - 2)} \delta_{L(0), \bar{L}(0)} \right) \quad (4.9c)$$

from the trace form of $Z(\tau, \bar{\tau})$ in the same equation. It remains to reconcile these conclusions with modular invariance in the full orbifold cosmological constant $\hat{\Lambda}$.

5 Provisional Forms of the Cosmological Constant

5.1 Derivation from the Spectral Equivalences

In this section, I will consider the following *one-parameter* β -family of candidates for the *full* orbifold cosmological constant

$$(\alpha'_c)^{13} \hat{\Lambda}^{(\lambda)}(\beta) = -\frac{1}{2} \int_F \frac{d^2\tau}{(\text{Im } \tau)^2} \left\{ (2\lambda - 1 - \beta(\lambda + 1)) Z(\tau, \bar{\tau}) + \right. \\ \left. + \beta \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \right\}. \quad (5.1)$$

of the permutation-orbifold-string system $U(1)^{26\lambda}/\mathbb{Z}_\lambda$ at $\hat{c} = 26\lambda$. The fundamental region F of moduli space is defined in Eq. (3.4). Note that the integrand of this expression is modular-invariant for each prime λ at any value of the arbitrary complex parameter β . Moreover, the integrand is independent of β when $\lambda = 1$, so that the provisional form (5.1) includes the ordinary cosmological constant $\Lambda = \hat{\Lambda}^{(1)}$ (see Eq. (3.4b)) of the untwisted closed string. It will be instructive to begin with the formal derivation of this family from the particle-theoretic contributions, returning in the following subsection to discuss special choices of β .

Collecting the twisted and untwisted particle-theoretic contributions in Eqs. (3.6a), (3.9) and (4.7), we find the naive, operator-sewn form of the full orbifold cosmological constant

$$\hat{\Lambda}_N^{(\lambda)} = -\frac{1}{2} \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} \left\{ \lambda Z(\tau, \bar{\tau}) + (\lambda - 1) \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right\} \quad (5.2)$$

where F_N is the naive integration region in Eq. (3.3c). In this form the integrand is not modular invariant, but in fact we can change the integrand

by the formal integration identities

$$\int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) = \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} \left\{ \alpha Z(\tau, \bar{\tau}) + \frac{\beta}{\lambda} Z(\lambda\tau, \lambda\bar{\tau}) \right\} \quad (5.3a)$$

$$\int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) = \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} \left\{ \gamma \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) + \delta Z(\tau, \bar{\tau}) \right\} \quad (5.3b)$$

$$\alpha + \beta = \gamma + \delta = 1 \quad (5.3c)$$

which follow straightforwardly from the formal spectral-equivalence relations (4.8) and (3.10a). Here α, β, γ and δ are new complex parameters constrained only by Eq. (5.3c). The further requirement $\gamma = \beta/(\lambda - 1)$ selects the following β -family of *modular-invariant* integrands

$$\begin{aligned} (\alpha'_c)^{13} \hat{\Lambda}_N^{(\lambda)} = & -\frac{1}{2} \int_{F_N} \frac{d^2\tau}{(\text{Im } \tau)^2} \left\{ (2\lambda - 1 - \beta(\lambda + 1)) Z(\tau, \bar{\tau}) + \right. \\ & \left. + \beta \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \right\} \quad (5.4) \end{aligned}$$

each of whose integrals is in fact equal to the β -independent naive form in Eq. (5.2). The final result (5.1) for the full physical orbifold cosmological constant then follows by the standard division $F_N \rightarrow F$ and $\hat{\Lambda}_N \rightarrow \hat{\Lambda}$ by $SL(2, \mathbb{Z})$.

This formal derivation emphasizes that the β -ambiguity in the physical result (5.1) is a consequence of the *formal spectral equivalences* discussed above, the entire β -family $\hat{\Lambda}(\beta)$ of provisional cosmological constants in Eq. (5.1) being rooted ($F \rightarrow F_N$) in the *same, β -independent sum (5.2) of particle-theoretic contributions* to the naive cosmological constant $\hat{\Lambda}_N$.

One may try to push the formal steps of this derivation a bit further: Using the spectral-equivalence identities (4.8) and (3.10a) again, one reconfirms that the final form (5.4) of the naive cosmological constant is indeed

independent of β

$$\begin{aligned}
(\alpha'_c)^{13} \frac{\partial \hat{\Lambda}_N^{(\lambda)}}{\partial \beta} &= \\
&= -\frac{1}{2} \int_{F_N} \frac{d^2 \tau}{(\text{Im } \tau)^2} \left\{ -(\lambda + 1) Z(\tau, \bar{\tau}) + Z(\lambda \tau, \lambda \bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right\} \\
&= 0
\end{aligned} \tag{5.5}$$

and then the modular invariance of this integrand apparently implies β -independence of the physical cosmological constant $0 = \infty \cdot (\partial \hat{\Lambda} / \partial \beta)$. This formal conclusion is thought-provoking, but I have been unable to verify it directly from the physical result (5.1) – and will not assume it in the discussion below.

5.2 Cases, Characters and So On

The one-parameter β -family $\hat{\Lambda}(\beta)$ of provisional forms (5.1) leaves us with un embarras de richesse, while each member of the family is itself unfamiliar. Since all the members of the family are formally associated to the same particle-theoretic contributions, it will be important to further test the provisional forms against other string-theoretic and conformal-field-theoretic intuitions. This includes in particular the β -dependent question whether any of these integrands have (or need to have) a *consistent interpretation at the level of characters* [7,13,35,36].

In this connection, I mention some special cases of $\hat{\Lambda}(\beta)$, starting with the case which is clearly favored by its simplicity:

$$\beta = 0 : \quad (\alpha'_c)^{13} \hat{\Lambda}^{(\lambda)} = -\frac{2\lambda - 1}{2} \int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) \tag{5.6a}$$

$$\hat{\Lambda}^{(\lambda)} = (2\lambda - 1) \Lambda. \tag{5.6b}$$

This form corresponds to the choices $\alpha = \delta = 1$ and $\gamma = 0$ in the derivation above, so that the formal integration identities in Eqs. (5.3a) and (5.3b) reduce to the primitive spectral equivalences (4.8) and (3.10a). Then it is clear that Eq. (5.6) shows the contributions of λ ordinary closed strings from the untwisted sector and $\lambda - 1$ ordinary closed strings from the twisted

sectors, and an ordinary untwisted character-theoretic interpretation of this case follows after compactification.

At the other extreme, I note the case $\beta = (2\lambda - 1)/(\lambda + 1)$

$$(\alpha'_c)^{13} \hat{\Lambda}^{(\lambda)} = -\frac{2\lambda - 1}{2(\lambda + 1)} \int_F \frac{d^2\tau}{(\text{Im } \tau)^2} \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \quad (5.7)$$

which shows only the Hecke transform of order λ . It is important to note that this form is also amenable to character analysis after compactification : Indeed, the existence of the modular-invariant Hecke transform tells us that even for the larger set of WZW cyclic permutation orbifold CFT's [13]

$$\frac{G^{(1)} \times \dots \times G^{(\lambda)}}{Z_\lambda}, \lambda \text{ prime} \quad (G \rightarrow U(1)^{26} \text{ for our abelian case}) \quad (5.8)$$

there must exist a hitherto-unnoticed *modular-covariant subset* of orbifold characters which corresponds to the Hecke form. Consulting Ref. [13], we recall that the so-called off-diagonal characters of the untwisted sector are responsible for the unwanted non-linear term Z^λ of the standard partition function – so the desired modular-covariant “Hecke-truncation” \mathcal{H} of the orbifold characters includes *all* the twisted characters but *only* the diagonal characters $\{\chi_p(\lambda\tau)\}$ of the untwisted sector.

The truncated subset \mathcal{H} can also be used to construct new non-abelian permutation orbifold CFT's beyond the standard prescription of Ref. [8]. We can be more explicit for the \mathbb{Z}_2 -permutation orbifolds $(G \times G)/\mathbb{Z}_2$, in

which case the closure of the subset \mathcal{H} under modular transformations

$$\chi_{(p-)}(\tau) \equiv (\chi_{(p0)} - \chi_{(p1)})(\tau) = \chi_p(2\tau), \quad Z(2\tau, 2\bar{\tau}) \sim \sum_p |\chi_p(2\tau)|^2 \quad (5.9a)$$

$$\mathcal{X}_{\widehat{(p\pm)}}(\tau) \equiv \left(\mathcal{X}_{\widehat{(p0)}} \pm \mathcal{X}_{\widehat{(p1)}} \right)(\tau) \quad (5.9b)$$

$$\chi_{(p-)}(\tau + 1) = \chi_{(p-)}(\tau), \quad \mathcal{X}_{\widehat{(p\pm)}}(\tau + 1) = T_p^{-\frac{1}{2}} \mathcal{X}_{\widehat{(p\mp)}}(\tau) \quad (5.9c)$$

$$\chi_{(p-)}\left(-\frac{1}{\tau}\right) = \sum_k S_{pk} \mathcal{X}_{\widehat{(k+)}}(\tau) \quad (5.9d)$$

$$\mathcal{X}_{\widehat{(p+)}}\left(-\frac{1}{\tau}\right) = \sum_k S_{pk} \mathcal{X}_{\widehat{(k-)}}(\tau) \quad (5.9e)$$

$$\mathcal{X}_{\widehat{(p-)}}\left(-\frac{1}{\tau}\right) = \sum_k P_{pk} \mathcal{X}_{\widehat{(k-)}}(\tau) \quad (5.9f)$$

is easily read from the modular transformations of the full set of orbifold characters in Ref. [13]. In this result, $\{\chi_p(2\tau)\}$ are the diagonal untwisted characters, $\{\mathcal{X}_{\widehat{(p\psi)}}(\tau)\}$ are the twisted characters, and the modular matrices T, S, P are defined in the (untwisted) mother theory G (see Eq. (5.8) and Ref. [13]).

For certain compactifications of the free-bosonic orbifolds at all prime λ , the work of Ref. [33] has previously identified a closely-related but not identical subset of characters which are closed under the modular subgroup $\Gamma_0(2)$ – but not closed under the modular group itself. In our notation, this $\Gamma_0(2)$ -covariant subset reduces for $H(\text{perm}) = \mathbb{Z}_2$ to the particular subset $\mathcal{S} = \{\chi_{(p-)}, \mathcal{X}_{\widehat{(p+)}}\}$ of our modular-covariant subset \mathcal{H} and, according to Eq. (5.9), the subset \mathcal{S} is closed under $\Gamma_0(2)$ for all the non-abelian orbifolds as well. Extensions of monster moonshine to $c = 24k$ have also been considered in Ref. [34].

Between the “elemental” cases in Eqs. (5.6) and (5.7), I also mention the

intermediate forms at $\beta = \lambda - 1$ and $(\lambda - 1)/\lambda$

$$(\alpha'_c)^{13} \hat{\Lambda}^{(\lambda)} = -\frac{1}{2} \int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} \left\{ \lambda(2 - \lambda) Z(\tau, \bar{\tau}) + \right. \\ \left. + (\lambda - 1) \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \right\} \quad (5.10a)$$

$$(\alpha'_c)^{13} \hat{\Lambda}^{(\lambda)} = -\frac{1}{2\lambda} \int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} \left\{ (\lambda^2 - \lambda + 1) Z(\tau, \bar{\tau}) + \right. \\ \left. + (\lambda - 1) \left[Z(\lambda\tau, \lambda\bar{\tau}) + \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) \right] \right\} \quad (5.10b)$$

the scale of whose Hecke terms is close in spirit to the standard CFT prescription [8]. Note that the special case $\lambda = 2$ of Eq. (5.10a)

$$(\alpha'_c)^{13} \hat{\Lambda}^{(2)} = -\frac{1}{2\lambda} \int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} \left[Z(2\tau, 2\bar{\tau}) + Z\left(\frac{\tau}{2}, \frac{\bar{\tau}}{2}\right) + Z\left(\frac{\tau+1}{2}, \frac{\bar{\tau}+1}{2}\right) \right] \quad (5.11)$$

coincides with the $\lambda = 2$ case of Eq. (5.7). The physical spectral equivalence given in Eq. (3.11) can of course be applied to the twisted contribution of any provisional form in (5.1), but the $\beta = \lambda - 1$ case in Eq. (5.10a) is the only one for which the scale of the twisted contribution

$$\hat{\Lambda}^{(\lambda)} = -\frac{\lambda - 1}{2} \int_F \frac{d^2 \tau}{(\text{Im } \tau)^2} \sum_{r=0}^{\lambda-1} Z\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}\right) = -\frac{\lambda - 1}{2} \int_{\hat{F}_\lambda} \frac{d^2 \tau}{(\text{Im } \tau)^2} Z(\tau, \bar{\tau}) \quad (5.12)$$

transparently shows the (differently cut off) contribution of $\lambda - 1$ ordinary closed strings. The new modular regions $\{\hat{F}_\lambda\}$ are defined in Eq. (3.11c). More generally, character analysis of intermediate cases such as those in Eq. (5.10) would apparently require an auxiliary set of ordinary untwisted characters for the $Z(\tau, \bar{\tau})$ term – in addition to the new modular subset \mathcal{H} of orbifold characters discussed above for the Hecke term.

6 One-Loop Diagrams with Insertions

6.1 Loops with a Cosmological Kernel

Following the principles used to obtain the provisional forms of the orbifold cosmological constant, I have also worked out a corresponding set of provisional one-loop orbifold-string diagrams with an arbitrary number of untwisted tachyonic *insertions* at $\alpha'_c k^2 = -2$. The twisted and untwisted contributions to loops of this type are depicted in Fig. 4, although the explicit labelling of the external lines in this figure corresponds to a later variant (see Eq. (6.13)) of the loops discussed here. For pedagogical purposes I will first present and discuss the final, physical form of these loops, postponing their formal particle-theoretic derivation until the latter part of this subsection.

The final result for this set $\{\hat{L}_n^{(\lambda)}(\{k\}, \beta)\}$ of loops with n insertions is structurally analogous to the orbifold cosmological constant $\hat{\Lambda}^{(\lambda)}(\beta)$:

$$(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\}, \beta) \equiv g^n \int_F \frac{d^2 \tau}{(2 \operatorname{Im} \tau)^2} \left\{ (2\lambda - 1 - \beta(\lambda + 1)) \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \right. \\ \left. + \beta \left[\mathcal{L}_n(\lambda \tau, \lambda \bar{\tau}; \{k\}) + \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \right] \right\} \quad (6.1a)$$

$$\mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \equiv (8\pi^2 \operatorname{Im} \tau)^{-12} |\eta(e^{2\pi i \tau})|^{-48} \times \\ \times (2i \operatorname{Im} \tau) \int_{G_\nu(\tau)} \left(\prod_{l=2}^n d^2 \nu_l \right) \prod_{i < j} \chi(\nu_{ij}, \tau, \bar{\tau})^{2\alpha'_c k_i \cdot k_j} \quad (6.1b)$$

$$\chi(\nu, \tau, \bar{\tau}) = 2\pi e^{-\pi \operatorname{Im}^2 \nu / \operatorname{Im} \tau} \left| \frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)} \right|, \quad \nu_{ij} = \nu_i - \nu_j \quad (6.1c)$$

$$G_\nu(\tau) : \quad |\operatorname{Re} \nu_i| \leq \frac{1}{2}, \quad \operatorname{Im} \tau \geq \operatorname{Im} \nu_i \geq 0, \quad i = 2, \dots, n. \quad (6.1d)$$

In this construction, F is the standard fundamental region and β is the same parameter that appears in the orbifold cosmological constant $\hat{\Lambda}^{(\lambda)}(\beta)$ in Eq. (5.1). The role of the ordinary partition function $Z(\tau, \bar{\tau})$ in the orbifold cosmological constant is now played by the modular-invariant densities $\{\mathcal{L}_n\}$ of the ordinary closed-string loops $L_n = L_n^{(1)}$, which are included in the result Eq. (6.1a) when $\lambda = 1$:

$$(\alpha'_c)^{13} \hat{L}_n^{(1)}(\{k\}) = g^n \int_F \frac{d^2\tau}{(2\text{Im}\tau)^2} \mathcal{L}_n(\tau, \bar{\tau}; \{k\}). \quad (6.2)$$

It follows that the β -dependent integrands in Eq. (6.1a) are modular-invariant for all prime λ .

I will call these provisional forms (6.1) the *loops with a cosmological kernel* because omission of the “insertion factor” in the second line of Eq. (6.1b) shows exactly the same provisional forms (5.1) of the orbifold cosmological constant

$$\hat{L}_n^{(\lambda)}(\{k\}, \beta) \rightarrow \hat{\Lambda}^{(\lambda)}(\beta) \quad (6.3)$$

as the “kernel” of each loop. This property is well-known (see e.f. Ref. [23]) for the ordinary closed-string loops at $\lambda = 1$, but we will see in the following subsection that the insertions and hence the kernel can be changed for the orbifolds with $\lambda \geq 2$.

Following the discussion in Sec. 5, I mention two special cases of the loops with a cosmological kernel:

$$\beta = 0 : \quad (\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\}) = g^n (2\lambda - 1) \int_F \frac{d^2\tau}{(2\text{Im}\tau)^2} \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \quad (6.4a)$$

$$\hat{L}_n^{(\lambda)}(\{k\}) = (2\lambda - 1) L_n(\{k\}) \quad (6.4b)$$

$$\begin{aligned} \beta = \lambda - 1 : \quad (\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\}) = g^n \int_F \frac{d^2\tau}{(2\text{Im}\tau)^2} \Big\{ & \lambda(2 - \lambda) \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \\ & + (\lambda - 1) \left[\mathcal{L}_n(\lambda\tau, \lambda\bar{\tau}; \{k\}) + \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \right] \Big\} \end{aligned} \quad (6.5a)$$

$$\int_F \frac{d^2\tau}{(2\text{Im}\tau)^2} \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) = \int_{\hat{F}_\lambda} \frac{d^2\tau}{(2\text{Im}\tau)^2} \mathcal{L}_n(\tau, \bar{\tau}; \{k\}). \quad (6.5b)$$

The provisional form in Eq. (6.4) is again the case favored by Occam’s razor – showing contributions to the orbifold loop of λ ordinary strings from the untwisted sector and $\lambda - 1$ ordinary strings from the twisted sectors. Beyond this case, the form in Eq. (6.5) is again the only case for which the scale of the twisted contribution to the physical loops transparently shows

the (differently cut off) contribution of $\lambda - 1$ ordinary strings – the physical spectral-equivalence identity in Eq. (6.5b) following by cut and paste as discussed in Subsec. 3.3.

I turn now to sketch the formal derivation of the orbifold loops (6.1) from their particle-theoretic contributions, which will also help to understand the insertions in these loops. For this discussion we will need the well-known *trace forms* of the ordinary closed-string loops, as they are obtained by sewing [34,12] of the ordinary closed-string trees:

$$(\alpha'_c)^{13} L_n(\{k\})_N = g^n \int (d^{26} J(0)) \{Tr(\gamma(k_n, 1, 1) D \dots D \gamma(k_1, 1, 1) + \dots\} \quad (6.6a)$$

$$= g^n \int_{F_N} \frac{d^2 \tau}{(2 \operatorname{Im} \tau)^2} \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \quad (6.6b)$$

$$\begin{aligned} \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) = & (2 \operatorname{Im} \tau)^2 \int (d^{26} J(0)) (2\pi)^n \int_{\tilde{G}_\nu(\tau)} \prod_{i=2}^n d^2 \nu_i \times \\ & \times \left\{ Tr(\gamma(k_n, 1, 1) \xi(\nu_n, \bar{\nu}_n; K_{n-1} - k) \gamma(k_{n-1}, 1, 1) \right. \\ & \cdot \xi(\nu_{n-1, n}, \bar{\nu}_{n-1, n}; K_{n-2} - k) \dots \gamma(k_1, 1, 1) \xi(\nu_2, \bar{\nu}_2; k) \\ & \cdot e^{-2\pi \operatorname{Im} \tau (L(0) + \bar{L}(0) - 2)} e^{2\pi i \operatorname{Re} \tau (L(0) - \bar{L}(0))}) + \dots \left. \right\} \end{aligned} \quad (6.6c)$$

$$\xi(\nu, \bar{\nu}; k) = e^{2\pi i (\nu L(0) - \bar{\nu} \bar{L}(0))}, \quad K_i = \sum_{j=1}^i k_j, \quad J(0) = \sqrt{\alpha'_c} k \quad (6.6d)$$

$$\tilde{G}_\nu(\tau) : \quad \operatorname{Im} \tau \geq \operatorname{Im} \nu_2 \geq \dots \geq \operatorname{Im} \nu_n \geq 0, \quad |\operatorname{Re} \nu_i| \leq \frac{1}{2}. \quad (6.6e)$$

This is the naive or *particle-theoretic* form of the ordinary loops, where F_N is the naive modular region in Eq. (3.3c), D is the untwisted propagator in Eq. (3.8c), $\gamma(k)$ is the ordinary closed-string vertex operator at $\alpha'_c k^2 = -2$ and the ellipses denote symmetrization with respect to the external momenta $\{k\}$.

Including the contributions of both the untwisted and twisted sectors, the corresponding trace form of the naive or *particle-theoretic* orbifold loop

is then easily written down

$$\begin{aligned}
(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\})_N = g^n \Big\{ \int (d^{26} J(0)) Tr(\tilde{\gamma}(k_n, 1, 1) \tilde{D} \dots \tilde{D} \tilde{\gamma}(k_1, 1, 1)) + \\
+ (\lambda - 1) \int (d^{26} \hat{J}(0)) Tr(\hat{g}(k_n, 1, 1) \hat{D} \dots \hat{D} \hat{g}(k_1, 1, 1) + \dots) \Big\}
\end{aligned} \tag{6.7a}$$

$$\tilde{\gamma}_{IJ}(k, \bar{z}, z) \equiv \gamma_I(k, \bar{z}, z) \delta_{IJ}, \quad I, J = 0, 1, \dots, \lambda - 1 \tag{6.7b}$$

$$\tilde{D}_{IJ} \equiv D[L_0^I(0), \bar{L}_0^I(0)] \delta_{IJ} \tag{6.7c}$$

$$\hat{g}(k, \bar{z}, z) \equiv \hat{g}\left(\mathcal{T} = tT \rightarrow \sqrt{\alpha'_c} k, \bar{z}, z\right), \quad \alpha'_c k^2 = -2. \tag{6.7d}$$

where D and \hat{D} are the untwisted and twisted propagators in Eq. (3.8c). In this expression, I have chosen the untwisted vertex operator $\tilde{\gamma}$ to describe untwisted tachyonic emission from the untwisted propagator \tilde{D} . Both structures $\tilde{\gamma}$ and \tilde{D} are $\lambda \times \lambda$ matrices, with a diagonal entry for each copy I in the untwisted sector. The trace in the first term of Eq. (6.7a) is defined to include the trace over $\{I, J\}$ space of the matrix product of these operators. For untwisted tachyonic emission from the twisted propagator \hat{D} , I have for simplicity chosen the reduced form \hat{g} of the twisted vertex operator, which masks the corresponding $\lambda \times \lambda$ matrix structure of the twisted vertex operator \hat{g} (see Eqs. (2.11) and (2.12)).

The trace-evaluated form of the sewn orbifold loop in Eq. (6.7) is as follows

$$\begin{aligned}
(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\})_N = \int_{F_N} \frac{d^2 \tau}{(2 \text{Im } \tau)^2} \Big\{ \lambda \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \\
+ (\lambda - 1) \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \Big\}
\end{aligned} \tag{6.8}$$

where $\{\mathcal{L}_n\}$ are the untwisted loop densities, and one also finds the formal integration identities

$$\int_{F_N} \frac{d^2 \tau}{(\text{Im } \tau)^2} \left\{ \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) - \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \right\} = 0 \tag{6.9a}$$

$$\int_{F_N} \frac{d^2 \tau}{(\text{Im } \tau)^2} \{ \mathcal{L}_n(\lambda \tau, \lambda \bar{\tau}; \{k\}) - \lambda \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \} = 0 \tag{6.9b}$$

which correspond respectively to the spectral equivalences in Eqs. (3.11a) and (4.8) for the orbifold cosmological constant. In particular, Eq. (6.9a) is a cut-and-paste identity (see Subsec. 3.3), and the identity in Eq. (6.9b) follows from the trace form (6.6c) of the densities $\{\mathcal{L}_n\}$ by steps analogous to those shown in Eq. (4.9).

I should emphasize that the untwisted vertex operator $\tilde{\gamma}$ in Eq. (6.7) couples identically to all the untwisted internal strings, which gives the factor λ in the first term of Eq. (6.8). This property is also noted in the Appendix, where the tree amplitudes of the vertex $\tilde{\gamma}$ are evaluated in terms of the ordinary closed-string trees. I have explicitly checked the twisted contribution of Eq. (6.7) to Eq. (6.8) only for the case $\lambda = 2$, but the result is certainly correct for all prime λ because Eq. (6.9a) shows that the trace-evaluated contribution of each twisted sector is equivalent to that of an ordinary closed string.

Thanks to the specific choice of emission vertices in Eq. (6.7), the reader will note that (up to a scale) all these trace-evaluated particle-theoretic results can be obtained by the simple map

$$Z(\tau, \bar{\tau}) \rightarrow \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \quad (6.10)$$

from the corresponding results above for the orbifold cosmological constant. It is then straightforward to write down the loop analogues of Eqs. (5.3) and (5.4) and divide by $SL(2, \mathbb{Z})$ to obtain the final result in Eq. (6.1) for the physical loops with a cosmological kernel.

6.2 Emissions of a Representative Tachyon

The orbifold-string loops (6.1) with a cosmological kernel were easy to understand in analogy with the orbifold cosmological constant but, as mentioned above, one can also construct modified orbifold loops which do not have a cosmological kernel when $\lambda \geq 2$.

I find it instructive to consider this topic first in a broader setting, beginning with the following more general set of manifestly modular-invariant

integrals:

$$(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\}; \beta, \rho, \eta) = g^n \int_F \frac{d^2 \tau}{(2 \operatorname{Im} \tau)^2} \left\{ A \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \right. \\ \left. + B \left[\mathcal{L}_n(\lambda \tau, \lambda \bar{\tau}; \{k\}) + \sum_{k=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \right] \right\} \quad (6.11a)$$

$$A = \rho \lambda + \eta(\lambda - 1) - \rho \beta(\lambda + 1), \quad B = \rho \beta. \quad (6.11b)$$

In this expression, $\{\mathcal{L}_n\}$ are the same ordinary, untwisted string densities and the parameter β is the same as above, but ρ and η are new complex parameters. Following the procedure outlined above for the loops with a cosmological kernel, these integrals can in fact be obtained by the map (6.10) and the corresponding loop-analogues of the spectral-equivalence identities (5.3) from the following naive form of the integrals

$$(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\}, \rho, \eta)_N = \int_{F_N} \frac{d^2 \tau}{(2 \operatorname{Im} \tau)^2} \left\{ \rho \lambda \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \right. \\ \left. + \eta(\lambda - 1) \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \right\}. \quad (6.12)$$

In particular, the parameter choice $\gamma = (\rho \beta / \eta(\lambda - 1))$ is necessary to obtain the modular-invariant integrands of Eq. (6.11) from the non-invariant integrands of Eq. (6.12). It is not clear that this naive form is always associated to an operator sewing, but we can be confident in certain cases: In the first place, we find for arbitrary values of the parameters and $\lambda = 1$ that both Eqs. (6.11) and (6.12) reduce to ρ times the ordinary parameter-independent untwisted string loops. Second, we recognize the case $\rho = \eta = 1$ as the physical and naive forms respectively of the orbifold-loops with a cosmological kernel (see Eqs. (6.1) and (6.8)).

Another case of particular interest is the parameter choice $\rho = 1/\lambda, \eta = 1$, which I will call the *orbifold loops of type I*. These are the I -independent loops for n emissions of a representative untwisted tachyon of type I :

$$(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\})_{NI} = g^n \int_{F_N} \frac{d^2 \tau}{(2 \operatorname{Im} \tau)^2} \left\{ \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \right. \\ \left. + (\lambda - 1) \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \right\} \quad (6.13a)$$

$$\begin{aligned} \rightarrow (\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\}, \beta)_I = g^n \int_F \frac{d^2\tau}{(2\text{Im}\tau)^2} \left\{ \left(\lambda + \frac{\beta}{\lambda}(\lambda+1) \right) \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) + \right. \\ \left. + \beta \left[\mathcal{L}_n(\lambda\tau, \lambda\bar{\tau}; \{k\}) + \sum_{r=0}^{\lambda-1} \mathcal{L}_n\left(\frac{\tau+r}{\lambda}, \frac{\bar{\tau}+r}{\lambda}; \{k\}\right) \right] \right\} \quad (6.13b) \end{aligned}$$

$$I = 0, 1, \dots, \lambda - 1 \quad (6.13c)$$

I have been careful to uniformize this notation – and the depiction of these loops in Fig. 4 – with the corresponding tree graphs $\{\hat{A}_{nI}^{(\lambda)}\}$ in Eq. (2.12) and Fig. 1. Comparing with the trace-evaluated result (6.8) for the naive loops with a cosmological kernel, we see that the internal twisted contribution to Eq. (6.13a) is here unchanged – but the kernel of the new loops is *not cosmological* for $\lambda \geq 2$ because the first term now shows only the contribution of a *single* untwisted internal closed string. The trace form of the naive loops $\{\hat{L}_n^{(\lambda)}(\{k\})_{nI}\}$ is then simply obtained by substitution of the type- I emission vertices (see Eqs. (2.11), (2.12) and (6.7b))

$$\tilde{\gamma} \rightarrow \gamma_I, \quad \hat{g} \rightarrow \hat{g}_I \quad (6.14)$$

in the trace form (6.7a) of the loop (and omission of the previously-implied trace over $\{I, J\}$ space in the untwisted term). Use of the type- I emission vertices does not change the internal contribution of the twisted sectors, but restricts the internal untwisted contribution to that of *type I alone* – thereby completing our identification of these loops. Because these loops and the corresponding trees of type I are independent of I , they may be used (in lieu of moding by permutations of $\{I\}$) at any fixed I to describe multiple emissions of a representative tachyon.

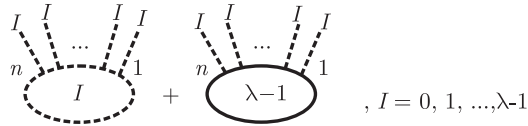


Fig. 4: Emissions of a representative tachyon.

In parallel to Eqs. (5.6) and (6.4), I finally mention the simplest case $\beta = 0$ of the physical loops (6.13b)

$$(\alpha'_c)^{13} \hat{L}_n^{(\lambda)}(\{k\})_I = g^n \lambda \int_F \frac{d^2\tau}{(2\text{Im}\tau)^2} \mathcal{L}_n(\tau, \bar{\tau}; \{k\}) \quad (6.15)$$

which clearly shows a single ordinary closed-string contribution from the untwisted sector and $\lambda - 1$ ordinary contributions from the twisted sectors.

7 Conclusions

Historically, conformal field theory and orbifold theory arose from string theory, and these five papers have outlined my ideas on closing the circle by constructing new physical string systems at higher central charge from the conformal field theory of orbifolds of permutation-type.

Many extensions of these ideas are indicated for future work. In particular, Ref. [31] and the present paper have so far considered only the very simplest of the new orbifold-string systems at the interacting level. These particular constructions allowed us to study the interacting systems in a context where, as we have seen, the physical spectra of the twisted strings are closely related to untwisted string spectra. Hopefully, these examples will then serve as prototypes for the study of the more general interacting orbifold-string systems of permutation-type, whose spectra are in fact generically-inequivalent [30] to untwisted systems.

Consider for example the *generalized* bosonic \mathbb{Z}_2 -permutation orbifolds $U(1)^{52}/H_+$ [28-30] with sectors labelled by elements of $H_+ = \mathbb{Z}_2 \times H$:

$$(1; \tau_+ \times \omega_2), \quad \omega_2^2 = 1 \quad (7.1a)$$

$$(1, \omega_3, \omega_3^2; \tau_+, \tau_+ \times \omega_3, \tau_+ \times \omega_3^2), \quad \omega_3^3 = 1 \quad (7.1b)$$

$$(1, \omega_4^2; \tau_+ \times \omega_4, \tau_+ \times \omega_4^3), \quad \omega_4^4 = 1. \quad (7.1c)$$

Here τ_+ permutes the two copies of untwisted $U(1)^{26}$, and $\{\omega_n\} \in H$ are extra automorphisms of order n which act uniformly on each copy. The low-order examples given here are easily extended to all n . All the sectors of this set of orbifolds are closed strings at $\hat{c} = 52$, the entries before and after the semicolon corresponding to untwisted and twisted sectors respectively. The extended Virasoro generators of all twisted open and closed $\hat{c} = 52$ strings were constructed in Ref. [30], and I remind that the $\hat{c} = 52$ string spectra are *not equivalent* to untwisted spectra when H has elements of order greater than two. This statement includes the twisted $\hat{c} = 52$ open-string spectra of the general bosonic orientation orbifolds $U(1)^{26}/(\mathbb{Z}_2(w.s.) \times H)$ [27-31].

One may further consider the generalized cyclic permutation orbifolds $U(1)^{26\lambda}/H_+$ [28,30] with $H_+ = \mathbb{Z}_\lambda \times H$ at $\hat{c} = 26\lambda$ for higher prime λ , all of whose twisted tree graphs should have the form in Eq. (2.9) with

the universal value (2.5b) of the quantity \hat{a}_λ in the twisted propagators. The twisted vertex operators for these orbifolds can be read as special cases of the general twisted bosonic vertex operators in Ref. [19]. For all these cases, an equivalent but *unconventionally-twisted* $c = 26$ description of the spectrum is easily written down using a simple variant (see Refs. [13,30] and Subsec. 2.3) of the *orbifold-induction procedure* with

$$J(M + \frac{\lambda n(r)}{\rho(\sigma)}) \equiv \hat{J}(m + \frac{n(r)}{\rho(\sigma)} + \frac{r}{\lambda}), \quad M \equiv \lambda m + r \quad (7.2a)$$

$$r = 0, 1, \dots, \lambda - 1, \quad n(r) \in (0, 1, \dots, \rho(\sigma) - 1). \quad (7.2b)$$

This map and the unhatted $c = 26$ modeling $(M + \frac{\lambda n(r)}{\rho(\sigma)})$ tells us that the physical spectra of these systems are again *inequivalent* to untwisted spectra cases unless $(\lambda n(r)/\rho(\sigma)) \in \mathbb{Z}$ for all “H-fractions” $\{n(r)/\rho(\sigma)\}$. (The quantities $\{n(r)\}$ and $\rho(\sigma)$ are the spectral indices [15,17,19,21,28] and order respectively of any element $\omega(\sigma)$ of H.)

Although I will not give the details here, one can in fact write down a generalized orbifold-induction procedure for every sector of the *generalized* permutation orbifold $U(1)^{26K}/(H(\text{perm}) \times H)$ [18,19,28] at $\hat{c} = 26K$, using a distinct map for each cycle in each element of the general permutation group $H(\text{perm})$ on K elements. On this basis, one sees that the physical spectra of these general systems are related to untwisted string spectra *only* for the special cases when $(f_j(\sigma)n(r)/\rho(\sigma)) \in \mathbb{Z}$ for all cycle lengths $\{f_j(\sigma)\}$ and H-fractions $\{n(r)/\rho(\sigma)\}$ of each sector. The simple subset of “ordinary” permutation orbifolds $U(1)^{26K}/H(\text{perm})$ at $\hat{c} = 26K$ satisfy these conditions with $(n(r)/\rho(\sigma)) = 1$, so (consistent with our work in this paper and in Refs. [28-30]) the physical spectra of all the corresponding “ordinary” permutation-orbifold strings are closely related to untwisted string spectra. Beyond these special cases, one expects [28-31] the vast majority of generalized permutation orbifolds to describe a great variety of new orbifold-string theories, yet to be studied at the interacting level.

Other directions, including partial orbifoldizations, partial compactifications, twisted B fields and the corresponding extensions to the superstring orbifolds of permutation-type are sketched in Ref. [28] and the introduction to this paper.

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A The Trees of the Vertex $\tilde{\gamma}$

The untwisted emission vertex $\tilde{\gamma}$ and propagator \tilde{D} are defined in Eqs. (6.7b) and (6.7c) of the text. Both structures are $\lambda \times \lambda$ matrices with a diagonal entry for each closed-string copy $I = 0, \dots, \lambda - 1$ in the untwisted sector of the orbifold. In accord with their loop contribution in Eq. (6.7a), the tree amplitudes $\{\tilde{A}_n^{(\lambda)}\}$ of the tilde system include a trace over matrix multiplication of these operators – which guarantees that the vertex $\tilde{\gamma}$ couples identically to all λ copies in the untwisted sector. This is clearly seen in the explicit evaluation of these trees

$$\widetilde{|0\rangle} \equiv |0\rangle_0 \otimes \dots \otimes |0\rangle_{\lambda-1} \quad (\text{A.1a})$$

$$\widetilde{|k\rangle}_{IJ} \equiv \lim_{z \rightarrow 0} \tilde{\gamma}_{IJ}(k, \bar{z}, z) \widetilde{|0\rangle} \quad (\text{A.1b})$$

$$= \delta_{IJ} |0\rangle_0 \otimes \dots \otimes |k\rangle_I \otimes \dots \otimes |0\rangle_{\lambda-1} \quad (\text{A.1c})$$

$$\tilde{A}_n^{(\lambda)}(\{k\}) \equiv \text{Tr}(\widetilde{\langle k_n |} \tilde{\gamma}(k_{n-1}, 1, 1) \tilde{D} \dots \tilde{D} \tilde{\gamma}(k_2, 1, 1) \widetilde{|k_1\rangle} + \dots) \quad (\text{A.1d})$$

$$= \sum_I \langle k_n | \gamma_I(k_{n-1}, 1, 1) D_I \dots D_I \gamma_I(k_2, 1, 1) |k_1\rangle_I + \dots \quad (\text{A.1e})$$

$$= \lambda A_n(\{k\}) = \text{Tr}(\mathbb{1}_\lambda \dots \mathbb{1}_\lambda) A_n(\{k\}) \quad (\text{A.1f})$$

$$A_n(\{k\}) = \tilde{A}_n^{(1)}(\{k\}) \quad (\text{A.1g})$$

$$= \langle k_n | \gamma(k_{n-1}, 1, 1) D \dots D \gamma(k_2, 1, 1) |k_1\rangle + \dots \quad (\text{A.1h})$$

where $\{A_n\}$ are the trees of the ordinary closed-string.

References

- [1] K. Bardakci and M. B. Halpern, “New dual quark models,” *Phys. Rev.* **D3** (1971) 2493.

- [2] M. B. Halpern, “The two faces of a dual pion-quark model,” *Phys. Rev.* **D4** (1971) 2398; R. Dashen and Y. Frishman, “Four-fermion interactions and scale invariance,” *Phys. Rev.* **D11** (1975) 2781; M. B. Halpern, “Quantum ‘solitons’ which are $SU(N)$ fermions,” *Phys. Rev.* **D12** (1975) 1684; V. G. Knizhnik and A. B. Zamolodchikov, “Current algebra and Wess-Zumino model in two dimensions,” *Nucl. Phys.* **B247** (1984) 83; M. B. Halpern and E. Kiritsis, “General Virasoro Construction on affine g ,” *Mod. Phys. Lett.* **A4** (1989) 1373; M. B. Halpern, E. Kiritsis, N. A. Obers and K. Clubok, “Irrational conformal field theory,” *Phys. Rep.* **265** (1996) 1, hep-th/9501144.
- [3] M. B. Halpern and C. B. Thorn, “The two faces of a dual pion-quark model II. Fermions and other things,” *Phys. Rev.* **D4** (1971) 3084.
- [4] E. Corrigan and D. B. Fairlie, “Off-shell states in dual resonance theory,” *Nucl. Phys.* **B91** (1975) 527; W. Siegel, “Strings with dimension-dependent intercept,” *Nucl. Phys.* **B109** (1976) 244.
- [5] J. Lepowsky and R. L. Wilson, “Construction of the affine Lie algebra $A_1^{(1)}$,” *Comm. Math. Phys.* **62** (1978) 43.
- [6] L. Dixon, J. Harvey, C. Vafa and E. Witten, “Strings on orbifolds,” *Nucl. Phys.* **B261** (1985) 678; “Strings on orbifolds II,” *Nucl. Phys.* **B274** (1986) 285. L. Dixon, D. Friedan, E. Martinec and S. Shenker, “The conformal field theory of orbifolds,” *Nucl. Phys.* **B282** (1987) 13. S. Hamidi and C. Vafa, “Interactions on orbifolds,” *Nucl. Phys.* **B279** (1987) 465. J. K. Freericks and M. B. Halpern, “Conformal deformation by the currents of affine g ,” *Ann. Phys.* **188** (1988) 258.
- [7] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, “The operator algebra of orbifold models,” *Comm. Math. Phys.* **123** (1989) 485.
- [8] A. Klemm and M. G. Schmidt, “Orbifolds by cyclic permutations of tensor-product conformal field theories,” *Phys. Lett.* **B245** (1990) 53.
- [9] G. Veneziano, “Construction of a crossing-symmetric Regge-behaved amplitude for linearly rising trajectories,” *Nuovo Cimento* **57A** (1968) 190.
- [10] M. A. Virasoro, “Alternative constructions of crossing-symmetric amplitudes with Regge behavior,” *Phys. Rev.* **177** (1969) 2309; J. Shapiro,

- “Electrostatic analogue for the Virasoro model,” *Phys. Lett.* **B33** (1970) 361.
- [11] S. Mandelstam “Dual resonance models,” *Phys. Rep.* **13** (1974) 259.
 - [12] M. B. Green, J. H. Schwarz and E. Witten, “Superstring theory,” Cambridge University Press, 1987.
 - [13] L. Borisov, M. B. Halpern, and C. Schweigert, “Systematic approach to cyclic orbifolds,” *Int. J. Mod. Phys.* **A13** (1998) 125, hep-th/9701061.
 - [14] J. Evslin, M. B. Halpern, and J. E. Wang, “General Virasoro construction on orbifold affine algebra,” *Int. J. Mod. Phys.* **A14** (1999) 4985, hep-th/9904105.
 - [15] J. de Boer, J. Evslin, M. B. Halpern, and J. E. Wang, “New duality transformations in orbifold theory,” *Int. J. Mod. Phys.* **A15** (2000) 1297, hep-th/9908187.
 - [16] J. Evslin, M. B. Halpern, and J. E. Wang, “Cyclic coset orbifolds,” *Int. J. Mod. Phys.* **A15** (2000) 3829, hep-th/9912084.
 - [17] M. B. Halpern and J. E. Wang, “More about all current-algebraic orbifolds,” *Int. J. Mod. Phys.* **A16** (2001) 97, hep-th/0005187.
 - [18] J. de Boer, M. B. Halpern, and N. A. Obers, “The operator algebra and twisted KZ equations of WZW orbifolds,” *J. High Energy Phys.* **10** (2001) 011, hep-th/0105305.
 - [19] M. B. Halpern and N. A. Obers, “Two large examples in orbifold theory: Abelian orbifolds and the charge conjugation orbifold on $su(n)$,” *Int. J. Mod. Phys.* **A17** (2002) 3897, hep-th/0203056.
 - [20] M. B. Halpern and F. Wagner, “The general coset orbifold action,” *Int. J. Mod. Phys.* **A18** (2003) 19, hep-th/0205143.
 - [21] M. B. Halpern and C. Helfgott, “Extended operator algebra and reducibility in the WZW permutation orbifolds,” *Int. J. Mod. Phys.* **A18** (2003) 1773, hep-th/0208087.

- [22] O. Ganor, M. B. Halpern, C. Helfgott and N. A. Obers, “The outer-automorphic WZW orbifolds on $so(2n)$, including five triality orbifolds on $so(8)$,” *J. High Energy Phys.* **0212** (2002) 019, hep-th/0211003.
- [23] J. de Boer, M. B. Halpern and C. Helfgott, “Twisted Einstein tensors and orbifold geometry,” *Int. J. Mod. Phys.* **A18** (2003) 3489, hep-th/0212275.
- [24] M. B. Halpern and C. Helfgott, “Twisted open strings from closed strings: The WZW orientation orbifolds,” *Int. J. Mod. Phys.* **A19** (2004) 2233, hep-th/0306014.
- [25] M. B. Halpern and C. Helfgott, “On the target-space geometry of the open-string orientation-orbifold sectors,” *Ann. of Phys.* **310** (2004) 302, hep-th/0309101.
- [26] M. B. Halpern and C. Helfgott, “A basic class of twisted open WZW strings,” *Int. J. Mod. Phys.* **A19** (2004) 3481, hep-th/0402108.
- [27] M. B. Halpern and C. Helfgott, “The general twisted open WZW string,” *Int. J. Mod. Phys.* **A20** (2005) 923, hep-th/0406003.
- [28] M. B. Halpern, “The orbifolds of permutation-type as physical string systems at multiples of $c = 26$: I. Extended actions and new twisted world-sheet gravities,” *J. High Energy Phys.* **0706** (2007) 068, hep-th/0703044.
- [29] M. B. Halpern, “The orbifolds of permutation-type as physical string systems at multiples of $c = 26$: II. The twisted BRST systems of $\hat{c} = 52$ matter,” *Int. J. Mod. Phys.* **A22** (2007) 4587, hep-th/0703208.
- [30] M. B. Halpern, “The orbifolds of permutation-type as physical string systems at multiples of $c = 26$: III. The spectra of $\hat{c} = 52$ strings,” *Nucl. Phys.* **B786** (2007) 297, hep-th/0704.1540.
- [31] M. B. Halpern, “The orbifolds of permutation-type as physical string systems at multiples of $c = 26$: IV. Orientation orbifolds include orientifolds,” *Phys. Rev.* **D76** (2007) 026004, hep-th/0704.3667.
- [32] R. Dijkgraaf, E. Verlinde and H. Verlinde, “Matrix string theory,” *Nucl. Phys.* **B500** (1997) 43, hep-th/9703030.

- [33] C. B. Thorn, “Embryonic dual model for pions and fermions,” *Phys. Rev.* **D4** (1971) 1112.
- [34] K. Bardakci, M. B. Halpern and J. Shapiro, “Unitary closed loops in Reggeized Feynman theory,” *Phys. Rev.* **185** (1969) 1910; D. Amati, C. Bouchiat and J. L. Gervais, “On the building of dual diagrams from unitarity,” *Nuovo Cimento Lett.* **2** (1969) 399; J. Shapiro, “Loop graph in the dual tube model,” *Phys. Rev.* **D5** (1972) 1945.
- [35] P. Bantay, “Characters and modular properties of permutation orbifolds,” *Phys. Lett.* **B419** (1998) 175, hep-th/9708120.
- [36] G. Cristofano, V. Marotta and G. Niccoli, “A new rational conformal field theory extension of the full degenerate $W_{1+\infty}^{(m)}$,” hep-th/0412085.
- [37] M. Jankiewicz and T. Kephart, “Extension of monster moonshine to $c = 24k$ conformal field theories,” hep-th/050178.